Milnor Fiber Consistency via Flatness

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Setup and Introduction

The Milnor Fibration

For a nonzero holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, we have the **Milnor fibration**

$$f : B_\varepsilon \cap f^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\}$$

for small enough $\varepsilon \gg \delta > 0$ (and it's independent of these choices).

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This is a locally trivial fibration whose fiber is a smooth manifold — the fiber's reduced homology is related somehow to the singularities of the hypersurface germ $(f^{-1}(0), 0)$.

The Question of Deformations

Suppose now that we have a holomorphic germ of a deformation of f given by

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}, 0),$$

where we think of $(\mathbb{C}^u, 0)$ as our (smooth) space of parameters.

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Question

Letting $\pi : \mathbb{C}^{n+1} \times \mathbb{C}^u \to \mathbb{C}^u$ be the projection and Δ the discriminant of $F \times \pi$, does

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap F^{-1}(D_\delta) \to D_\delta \times B_\gamma$$

define a smooth locally trivial fibration over the complement of Δ for small enough $\varepsilon \gg \delta, \gamma > 0$?

Elementary Facts

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- If $f^{-1}(0)$ has an isolated singularity at the origin, then the answer is always **yes**.
- This lets us completely understand the homology of the Milnor fiber in the isolated case by perturbing our function slightly to break the critical locus up into Morse points. (More on this in a second!)
- In general, the answer is **no**; consider $F((x, y, z), t) = xy - tz$.

Prior Work

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- In particular, Bobadilla has a theory of "morsification relative to an ideal" which gives circumstances in which we can hold the positive-dimensional parts of the critical locus fixed and perturb the zero-dimensional stuff to split off some Morse points.
- Massey has invariants called Lê numbers whose constancy at the origin in a family implies the consistency (in general, homologically rather than diffeomorphically) of the Milnor fiber; however, this requirement is too stringent to account for these kinds of splitting behaviors.

Results Using Scheme Structure

We can give a more comprehensive condition under which the Milnor fiber varies consistently using algebra:

Theorem

The answer to our question is **yes** *as long as the scheme-theoretic critical locus of $F \times \pi$ is flat over the parameter space $\mathbb{C}^u$ at the origin — that is, so long as the natural map of convergent power series rings*

$$\mathbb{C}\{t_1, \ldots, t_u\} \to \frac{\mathbb{C}\{x_0, \ldots, x_n, t_1, \ldots, t_u\}}{\left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}\right)}$$

is flat.

Motivation and Algebraic Background

The Milnor Number

In the case where f defines an isolated singularity, the homology of the Milnor fiber is determined completely by the non-reduced structure of the critical locus — in particular, by the **Milnor number**

$$\mu_f := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, \ldots, x_n\}}{\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right)}.$$

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If we look at $f_t = f + tg$ for g a generic linear form, we see that, for small t, f_t has μ_f Morse critical points near the origin, each of which contributes a single $\mathbb{Z}$-summand to the nth reduced homology of the smooth fiber:

$$\tilde{H}_k(F_f; \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\oplus \mu_f} & k = n \\ 0 & k \neq n \end{cases}$$

The Milnor Number (cont.)

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To answer this, we want to be able to say that the critical locus of f_t remains consistent as we vary t in some way that respects the non-reduced structure. This is precisely the algebro-geometric notion of **flatness**.

Flatness

- We say that a map $\phi : X \to Y$ of schemes or complex-analytic spaces is **flat** at $x \in X$ if $- \otimes_{\mathcal{O}_{Y,\phi(x)}} \mathcal{O}_{X,x}$ is an exact functor.

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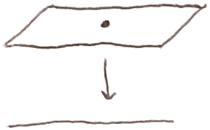
- If Y is smooth and one-dimensional, flatness is the same as the condition that no component (irreducible or embedded) be mapped to a single point. (A **component** is a closed subset which is an irreducible component of the support of some section of the structure sheaf.)

Flatness (cont.)

Non-Example

The map $\mathbb{C}\{x\} \to \mathbb{C}\{x, y, z\}/(xz, yz, z^2)$ *is not flat.*



Flatness (cont.)

Example

The map $\mathbb{C}\{x\} \to \mathbb{C}\{x, y, z\}/(yz, (z - x^2)z)$ *is flat.*

The Isolated Case

Proposition

If F defines an isolated singularity at the origin, then the map

$$\mathbb{C}\{t_1, \ldots, t_u\} \to \frac{\mathbb{C}\{x_0, \ldots, x_n, t_1, \ldots, t_u\}}{\left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}\right)}$$

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is automatically flat.

This explains why the Milnor number is the same as the number of Morse points we get on perturbation — for isolated singularities both the flatness condition and the consistency of the Milnor fiber hold without additional hypotheses.

Main Theorem and Consequences

Theorem (again)

Given a holomorphic germ of a deformation of a hypersurface singularity

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}, 0)$$

such that the critical locus varies consistently in the sense that

$$\mathbb{C}\{t_1, \ldots, t_u\} \to \frac{\mathbb{C}\{x_0, \ldots, x_n, t_1, \ldots, t_u\}}{(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})}$$

is flat, the Milnor fiber varies consistently in the sense that

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap F^{-1}(D_\delta) \to D_\delta \times B_\gamma$$

defines a smooth locally trivial fibration over the complement of the discriminant for small enough $\varepsilon \gg \delta, \gamma > 0$.

Proof idea

Using Thom's first isotopy lemma, we see that the main challenge is to produce evidence that the smooth fibers of $F \times \pi$ are transverse to the boundary sphere $S_\varepsilon \times \mathbb{C}^u$.

This can be accomplished by a lifting of vector fields tangent to the fibers of f — that is, it works because the flatness of the critical locus of $F \times \pi$ is equivalent to the property that any vector field germ satisfying $Wf = 0$ can be extended to a family $\tilde{W}$ satisfying $\tilde{W}F = 0$.

Corollary (Scheme Structure Sees Fiber Changes)

Let f be a holomorphic function and C the critical locus of f. If C_{red} is smooth and C has no embedded components, then the diffeomorphism type of the (transversal) Milnor fiber is locally constant along C.

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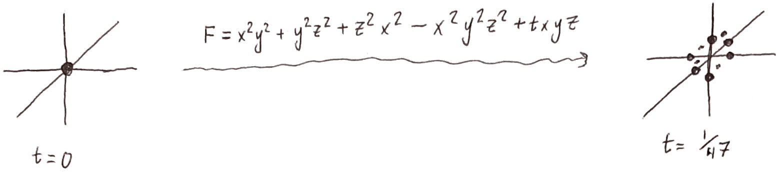
Construct a deformation by translations of f (or its restriction to a transversal slice) with parameter space C_{red}.

Milnor Fiber Homology Through Deformations

Example (Maxim-Rodriguez-Wang)

Let $f = x^2y^2 + y^2z^2 + z^2x^2 - x^2y^2z^2$. *We use a sequence of deformations to compute the reduced homology of the Milnor fiber.*

First we pull the fuzz away from the origin:



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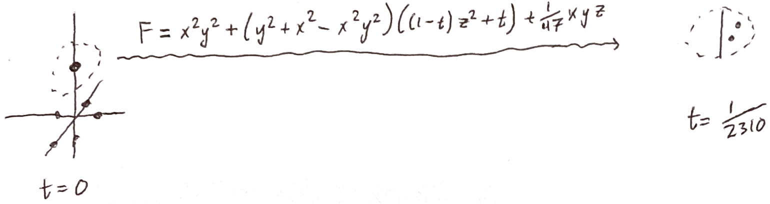
Now we separate each of the six double points from the axes:



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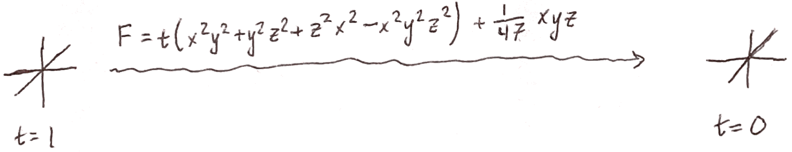
Finally, we get the fiber at the origin in a form that's easy to compute:



Milnor Fiber Homology Through Deformations (cont.)

So, in summary, for $f = x^2y^2 + y^2z^2 + z^2x^2 - x^2y^2z^2$, we can use the main theorem to pull 16 Morse points out of the critical locus, each of which contributes a vanishing cycle in degree 2. What's left over can be deformed into xyz, whose Milnor fiber is well-known to be a torus $(\mathbb{C}^*)^2$ — hence, for our original f, we can apply results of Siersma to get a direct sum decomposition

$$\tilde{H}_k(F_f; \mathbb{Z}) = \begin{cases} 0 & k = 0 \\ \mathbb{Z}^{\oplus 2} & k = 1 \\ \mathbb{Z}^{\oplus 17} & k = 2 \\ 0 & k > 2. \end{cases}$$

Conclusion

A Note on the Converse

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Is the flatness condition necessary as well as sufficient for the Milnor fiber to vary consistently?

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You may have noticed that the proof will actually work whenever failure of flatness occurs only at the origin — that is, when

$$\dim_{\mathbb{C}} \operatorname{Tor}^{\mathbb{C}\{t_1,\ldots,t_u\}}\left(\mathbb{C}, \frac{\mathbb{C}\{x_0,\ldots,x_n,t_1,\ldots,t_u\}}{\left(\frac{\partial F}{\partial x_0},\ldots,\frac{\partial F}{\partial x_n}\right)}\right) < \infty.$$

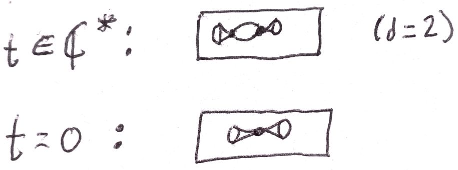
The Converse is False

Counterexample (Bobadilla)

Consider the family given by

$$F((x_1, x_2, x_3, y_1, y_2), t) = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} x_3 & x_2 \\ x_2 & x_3 - x_1{}^d + tx_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then the Tor module from the previous slide has dimension $d - 1$ (at least for $d \leq 5$).



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- Generalize in various directions (CIS, real-analytic setting, singular spaces, etc.)

Thanks for Listening!