Milnor Fiber Consistency for Deformations of Arbitrarily-Singular Hypersurfaces

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July 20, 2022

Review: The Milnor Fibration

- Given a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, we take its restriction to $B_\varepsilon \cap f^{-1}(D_\delta)$ for $1 \gg \varepsilon \gg \delta > 0$; throwing away the potentially singular fiber over the origin, we get a smooth locally trivial **Milnor fibration** over D^*_δ. Denote its fiber by F_f.

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- The homological structure of F_f is related somehow to the singular structure of $(f^{-1}(0), 0)$.
- E.g., **Milnor-Kato-Matsumoto:** $\tilde{H}_i(F_f) = 0$ for $i \notin [n - \dim_{\mathbb{C}} \mathrm{Sing}_0(f^{-1}(0)), n]$

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- (D. Massey) When the perturbation **preserves the Lê numbers** at 0 — this is too rigid for the kind of splitting we have in the isolated case.
- (J. F. de Bobadilla, R. Pellikaan, ...) When the perturbation is through an ideal with respect to which f has **finite extended codimension** — this can split off isolated singularities, but requires the higher-dimensional components to be fixed.

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Theorem

Suppose $C_{F\times\pi}$ is flat over $\mathbb{C}^u$ at the origin. Then

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap F^{-1}(D_\delta) \to D_\delta \times B_\gamma$$

defines a smooth locally trivial fibration over the complement of $\bar{\Delta}$ for $1 \gg \varepsilon \gg \delta, \gamma > 0$.

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- However, if we account for the nonreduced behavior — that is, consider it as a "point with infinitesimal fuzz" — we see that it is just the spectrum of the **Milnor algebra** $\mathcal{O}_{\mathbb{C}^{n+1},0}/(\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n})$ and hence determines the homology of F_f completely.

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- By analogy, we'll consider

$$C_{F\times\pi} = V(\frac{\partial F}{\partial x_0},\ldots,\frac{\partial F}{\partial x_n}) = \operatorname{Specan} \mathcal{O}_{\mathbb{C}^{n+1}\times\mathbb{C}^u,0}/(\frac{\partial F}{\partial x_0},\ldots,\frac{\partial F}{\partial x_n})$$

as an *analytic scheme* ($\mathbb{C}$-analytic space).

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- So, for us, the flatness requirement is to say that the natural map

$$\mathbb{C}\{t_1, \ldots, t_u\} \to \frac{\mathbb{C}\{x_0, \ldots, x_n, t_1, \ldots, t_u\}}{(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})}$$

of convergent power series rings is flat.

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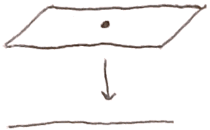
- (H. Hironaka(?)) Flatness at a point is equivalent to the local triviality of the normal cone to the fiber over the tangent cone of the base.
- If Y is one-dimensional and smooth, then flatness is equivalent to the requirement that no component, irreducible or embedded, be mapped to a single point.

A Non-Flat Map

Non-Example

The map $\mathbb{C}\{x\} \to \mathbb{C}\{x,y,z\}/(xz, yz, z^2)$ *is not flat.*



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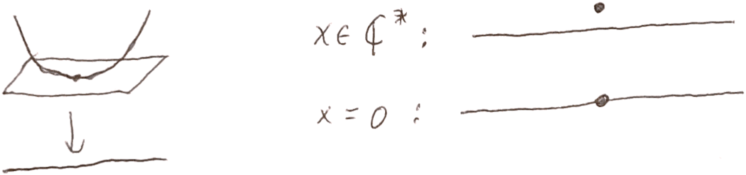

Aside on the Isolated Case

- If f defines an isolated singularity, then $C_{F \times \pi}$ will *always* be flat over $\mathbb{C}^u$.
- This explains why the algebraic Milnor number is the same as the Morse-theoretic one — the flatness forces the consistency of number of points in the fiber when counted with multiplicity.

Interpretation of the Theorem

Theorem (again)

Suppose $C_{F\times\pi}$ is flat over $\mathbb{C}^u$ at the origin. Then

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defines a smooth locally trivial fibration over the complement of $\bar{\Delta}$ for $1 \gg \varepsilon \gg \delta, \gamma > 0$.

That is: If the critical loci of a family of holomorphic function germs vary consistently (in the sense of flatness), then their local smooth fibers vary consistently as well (in the sense that they fit together into a smooth locally trivial fibration).

Example Use

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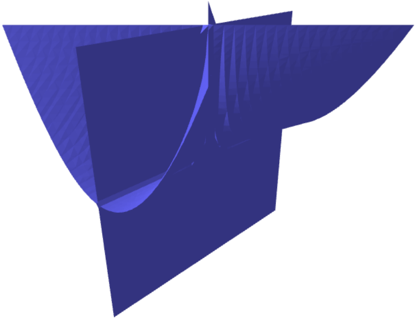
Let $f = x^3 + xy^2z$. *We compute the reduced homology of the Milnor fiber by the deformation* $F = (x^2 + y^2z - 5t^2)(x - t)$.



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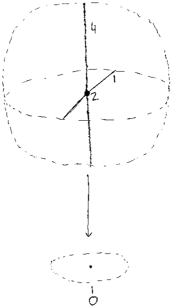

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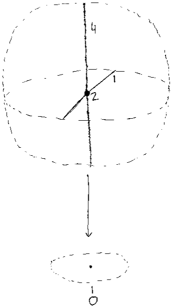
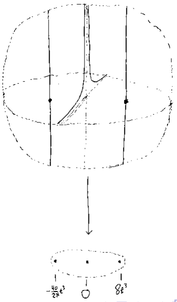
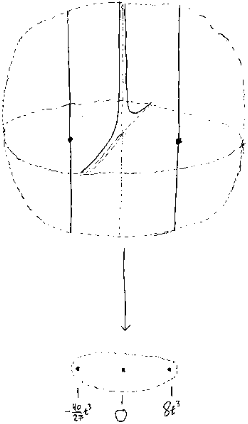




Computing Homology



- (D. Siersma) Express the Milnor fiber homology as the (shifted) relative homology of $({f_t}^{-1}(D_\delta), {f_t}^{-1}(v))$.

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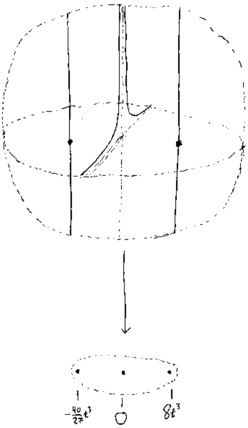

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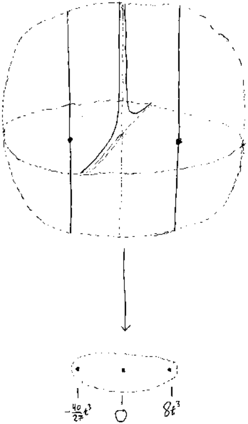

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- (D. Siersma) This can be computed locally at each critical value.
- In this case the bits above $-\frac{40}{27}t^3$ and $8t^3$ are D_∞ singularities, so each contributes a $\mathbb{Z}$-summand in degree 2.

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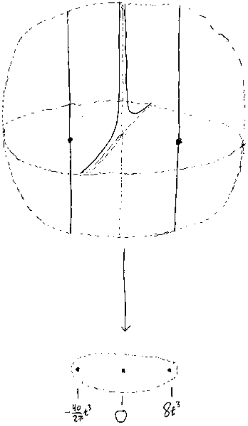

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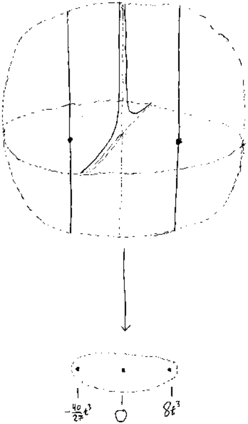

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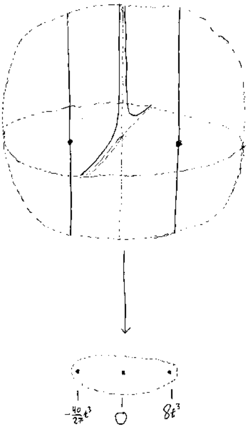

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- The nearby fiber is thus homotopic to an S^1 bundle over $V(x - t, y^2 z - 4t^2) \simeq S^1$.
- We see that this bundle has trivial monodromy, so it is actually a torus and hence the *relative* homology gives $\mathbb{Z}$-summands in degrees 1 and 2.

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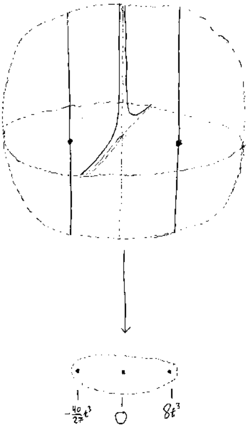

Therefore we arrive at the homology of the original Milnor fiber:

$$\tilde{H}_i(F_f) \cong \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \\ \mathbb{Z}^{\oplus 3} & i = 2 \\ 0 & i > 3 \end{cases}$$

Questions for the Future

- Given f, how can we find interesting deformations satisfying the criterion?
- Is there a way to see changes in the (transversal?) Milnor fiber from the scheme structure of C_f? For example, is there a way to produce a Whitney stratification of $f^{-1}(0)$ from this info?
- Less plausibly: Is there a way to read the homology of the Milnor fiber directly from scheme-theoretic invariants of C_f?
- Does this perspective generalize to other settings? (CIS, real-analytic singularities, holomorphic functions on a singular space, etc.)

Thanks for listening!