Milnor Fiber Consistency via Flatness

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Abstract

The Milnor fiber of a holomorphic function defining an isolated singularity can be understood by perturbing the function slightly to one with only Morse critical points. For an arbitrary-dimensional singularity, perturbation is no longer guaranteed to preserve the Milnor fiber, making an analogous approach difficult. We present a new algebraic condition — the flatness of the critical locus over the parameter space — under which this problem does not arise, giving a new avenue for Milnor fiber computations.

Background and Motivation

Milnor proved in [Mil68] that, if f defines an isolated singularity at the origin, then $\mathbb{F}_f \simeq \bigvee_{\mu_f} S^n$, where $\mu_f := \dim_{\mathbb{C}} \dfrac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{\left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}\right)}$.

This is to say that the homotopy type of the Milnor fiber is determined entirely by the critical locus of f, *provided* we endow it with the nonreduced structure given by the **Jacobian ideal** $J_f := \left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}\right)$. This naturally leads us to ask:

Question What does the critical locus of f (with the non-reduced structure given by J_f) tell us about $\mathbb{F}_f$ in the non-isolated case?

Main Result

Theorem (H.) Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}, 0)$ be a germ of a family of holomorphic functions. Suppose the critical loci of the functions in the family, as given by $J_{F\times\pi} := \left(\frac{\partial F}{\partial x_0}, \cdots, \frac{\partial F}{\partial x_n}\right)$, are flat over $(\mathbb{C}^u, 0)$ away from 0×0 on $(\mathbb{C}^{n+1} \times 0, 0 \times 0)$. Then

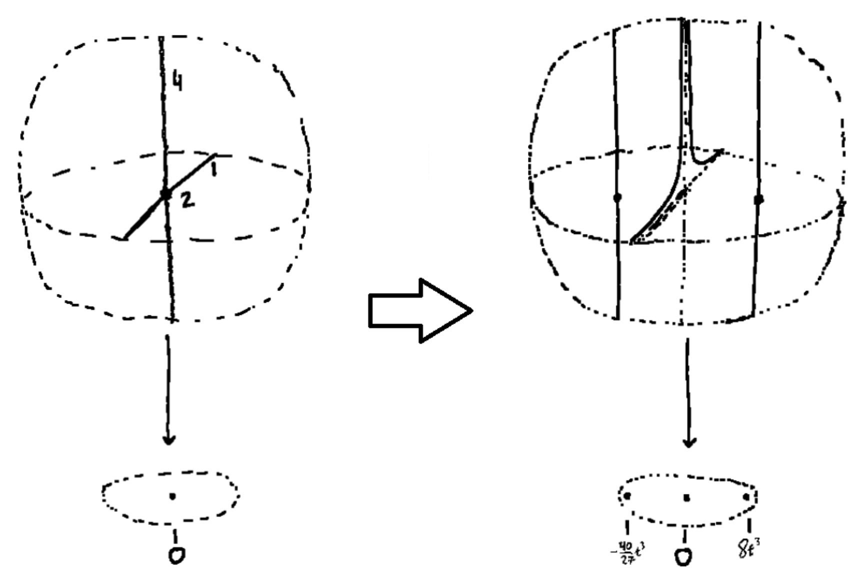
$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap F^{-1}(D_\delta) \to D_\delta \times B_\gamma$$

is a fibration away from the discriminant for $\pi : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}^u, 0)$ the projection and $1 \gg \varepsilon \gg \delta, \gamma > 0$.

This tells us that the scheme-theoretic consistency of the critical loci implies consistency of the smooth fibers.

An Example Deformation

Let $f(x, y, z) = x^3 + xy^2z$. Then $F(x, y, z, t) = (x^2 + y^2z - 5t^2)(x - t)$ deforms the critical locus flatly, splitting off two D_∞ singularities:



We can then find that $\tilde{H}_1(\mathbb{F}_f) \cong \mathbb{Z}$, $\tilde{H}_2(\mathbb{F}_f) \cong \mathbb{Z}^{\oplus 3}$, and all other homology groups are zero.

Homogeneous Polynomials

For homogeneous polynomials, the situation is particularly nice:

Corollary (H.) Fix $n \geq 0$ and $d \geq 1$, and let $H_{n,d} \cong \mathbb{P}^{\binom{n+d}{n}-1}$ be the space of nonzero degree-d homogeneous polynomials in $x_0, \ldots, x_n$ up to scaling. Let $C_{n,d} \subset \mathbb{P}^n \times H_{n,d}$ be the projective vanishing of the corresponding **universal Jacobian ideal** $J_{n,d}$ given by the partial derivatives with respect to the x_i of the universal polynomial $\sum_{|\alpha|=d} a_\alpha \vec{x}^\alpha$. Then the **flattening stratification** of the natural projective morphism $C_{n,d} \to H_{n,d}$ is such that the diffeomorphism types of the (affine) Milnor fibers of the polynomials corresponding to points of $H_{n,d}$ are constant along each stratum, as are the monodromy diffeomorphisms.

Here "flattening stratification" is meant in the sense of [Gro62; Fan+05].

Toy Example: Conics

Let $a_{ij}, 0 \leq i \leq j \leq n$ be the coordinates of $H_{n,2} \cong \mathbb{P}^{\frac{n(n+3)}{2}}$, so that the universal polynomial is given by $\sum_{0 \leq i \leq j \leq n} a_{ij} x_i x_j$. Then the Corollary tells us that the polynomials' smooth fibers and monodromies are locally constant along the rank-r locus of the matrix

$$\begin{bmatrix} 2a_{00} & a_{01} & \cdots & a_{0n} \\ a_{01} & 2a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{0n} & a_{1n} & \cdots & 2a_{nn} \end{bmatrix}$$

in $H_{n,2}$ for each $1 \leq r \leq n$.

Harder Example: Plane Cubics

Let $a_0, a_1, a_2, b_{01}, b_{02}, b_{10}, b_{12}, b_{20}, b_{21}, c$ be the coordinates of $H_{2,3} \cong \mathbb{P}^9$, so that the universal polynomial is given by $a_0x^3 + a_1y^3 + a_2z^2 + b_{01}x^2y + b_{02}x^2z + b_{10}y^2x + b_{12}y^2z + b_{20}z^2x + b_{21}z^2y + cxyz$. Then, using results from [Gro62; Fan+05], the polynomials' smooth fibers and monodromies are locally constant along the strata of the stratification induced by all the Fitting ideals of the $\mathbb{C}[a_0, a_1, a_2, b_{01}, b_{02}, b_{10}, b_{12}, b_{20}, b_{21}, c]$-modules

$$\frac{(x, y, z)^{N+i}}{J_{2,3} \cap (x, y, z)^{N+i} + (x, y, z)^{N+i+1}}$$

for $N \gg 0$ and $0 \leq i \leq 2$.

References Cited

[Fan+05] Barabara Fantechi et al. *Fundamental Algebraic Geometry: Grothendieck's FGA Explained.* Vol. 123. Mathematical Surveys and Monographs. American Mathematical Society, 2005. ISBN: 9780821842454.

[Gro62] Alexander Grothendieck. *Fondements de la Géométrie Algébrique.* Extraits du Séminaire Bourbaki, 1957-1962. Paris: Secrétariat Mathématique, 1962.

[Mil68] John Milnor. *Singular Points of Complex Hypersurfaces.* Princeton, New Jersey: Princeton University Press and the University of Tokyo Press, 1968.