Milnor Fiber Consistency via Flatness

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Singularities in the Midwest VIII
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Example: $f(x, y) = xy$

- One level set of this function, $f^{-1}(0)$, is special.
- The rest pretty much all look the same.

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- $f|_{\mathbb{C}^{n+1} \setminus V(f)}$ is a smooth locally trivial fibration over $\mathbb{C}^*$, called the **affine Milnor fibration** of f — its fibers $f^{-1}(\lambda), \lambda \neq 0$ are manifolds, but may have nontrivial topological structure.

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Very little is known about the topology of $f^{-1}(\lambda)$, even in situations where f is nice!

Motivating Question

We want to know how the affine Milnor fibration relates to the structure of $V(f)$. In particular:

Question

If $f_{t_1,\ldots,t_u}$ is a family of degree-d homogeneous polynomials, when will the smooth fiber ${f_{t_1,\ldots,t_u}}^{-1}(\lambda)$ be independent of the parameters $t_1,\ldots,t_u$?

Local Version:
The Milnor Fibration

Example: $f(x, y) = x(x-1)y(y-1)$

- Since this polynomial is not homogeneous, there is no longer *necessarily* anything special about the fiber over the origin.
- In this case, $f^{-1}(0)$ is no longer contractible.
- However, if we zoom in, we get a picture a lot like we had before.

Definition of the Milnor Fibration

Definition (Milnor '68, Lê '73)

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a holomorphic function germ. Its restriction*

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If f is a homogeneous polynomial, its Milnor fibration at the origin is diffeomorphically equivalent to its affine Milnor fibration.

Relationship with the Critical Locus

- (Kato-Matsumoto '75) Let s be the dimension of the singular locus of $V(f)$ (i.e., of the critical locus of f) at 0. Then $\tilde{H}_i(\mathbb{F}_f) = 0$ for all $i \notin [n-s, n]$.

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- (Milnor '68) Suppose $s = 0$. Then, for

$$\mu_f := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{\left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}\right)},$$

we have $\mathbb{F}_f \simeq \bigvee_{\mu_f} S^n$ (and so $\tilde{H}_n(\mathbb{F}_f) \cong \mathbb{Z}^{\oplus \mu_f}$).

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we have $\mathbb{F}_f \simeq \bigvee_{\mu_f} S^n$ (and so $\tilde{H}_n(\mathbb{F}_f) \cong \mathbb{Z}^{\oplus \mu_f}$). That is, the homotopy type of the Milnor fiber is determined entirely by the nonreduced structure of $C_f := V(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$.

Main Idea

It seems plausible that the same should be true in the non-isolated case:

Idea

Let $J_f := (\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$ and $C_f := V(J_f)$. Then the Milnor fibration of f should be determined by the inclusion $C_f \hookrightarrow \mathbb{C}^{n+1}$ of the critical locus into the ambient space — that is, by the algebraic invariants of the quotient map

$$\mathbb{C}\{x_0, \ldots, x_n\} \twoheadrightarrow \mathbb{C}\{x_0, \ldots, x_n\}/J_f .$$

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We will prove a relative version of this idea.

Families of Functions

The Question of Consistency

Suppose now that we have a holomorphic germ of a deformation of f given by

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}, 0),$$

where we think of $(\mathbb{C}^u, 0)$ as our (smooth) space of parameters.

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Question

Letting $\pi : \mathbb{C}^{n+1} \times \mathbb{C}^u \to \mathbb{C}^u$ be the projection and Δ the discriminant of $F \times \pi$, does

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap F^{-1}(D_\delta) \to D_\delta \times B_\gamma$$

define a smooth locally trivial fibration over the complement of $\bar{\Delta}$ for $1 \gg \varepsilon \gg \delta, \gamma > 0$?

Some Answers

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- In general, the answer is **no**; consider $F((x, y, z), t) = xy - tz$.
- (Siersma '83, . . . , Bobadilla '05) The answer is **yes** if we deform through an ideal with respect to which f has **finite extended codimension**.
- (Massey '90-91) The answer is **yes** if the **Lê numbers** at the origin are constant in the family.

Relative Version of the Main Idea

Idea

Let $J_{F\times\pi} := (\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})$ *and* $C_{F\times\pi} := V(J_{F\times\pi})$. *Then the answer to our consistency question should be* **yes** *so long as the embedding* $C_{F\times\pi} \hookrightarrow \mathbb{C}^{n+1} \times \mathbb{C}^u$ *satisfies some algebro-geometric notion of consistency over* $\mathbb{C}^u$.

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- If we just want consistency for $C_{F\times\pi}$ itself, the correct requirement is **flatness** over $\mathbb{C}^u$. (This will turn out to be enough if f is well-behaved.)
- If we want the *embedding* to be consistent, we should ask for all of the **infinitesimal neighborhoods** $V({J_{F\times\pi}}^{k+1})$, $k \geq 0$ to be flat over $\mathbb{C}^u$ — or, equivalently, for the **normal cone**

$$C_{C_{F\times\pi}}(\mathbb{C}^{n+1}\times\mathbb{C}^u) := \operatorname{Spec}\operatorname{gr}_{J_{F\times\pi}}\mathbb{C}\{x_0,\dots,x_n,t_1,\dots,t_u\}$$
$$:= \operatorname{Spec}\bigoplus_{k\geq 0}{J_{F\times\pi}}^k/{J_{F\times\pi}}^{k+1}$$

to be so.

Main Theorem

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Theorem (H. '23)

Suppose that either of the following conditions holds:

- *For every critical point $p \neq 0$ of f near the origin, there exists a $V \in \Theta_{\mathbb{C}^{n+1} \times 0, p}$ such that $Vf = 0$ and $((\sum_{i=0}^{n} \bar{x}_i dx_i) \cdot V)(p) \neq 0$. Moreover, $C_{F \times \pi}$ is flat over $\mathbb{C}^u$ everywhere on $\mathbb{C}^{n+1} \times 0$, except possibly at the origin.*
- *$C_{C_{F \times \pi}}(\mathbb{C}^{n+1} \times \mathbb{C}^u)$ is flat over $\mathbb{C}^u$ everywhere on $\mathbb{C}^{n+1} \times 0$, except possibly at the origin.*

Then, for $1 \gg \varepsilon \gg \delta, \gamma > 0$,

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap F^{-1}(D_\delta) \to D_\delta \times B_\gamma$$

defines a smooth locally trivial fibration over the complement of $\bar{\Delta}$.

Comparisons with Other Answers

- If f defines an isolated singularity, then the normal cone to $C_{F\times\pi}$ is automatically flat over $\mathbb{C}^u$ — even at the origin!

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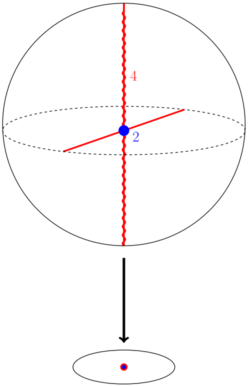
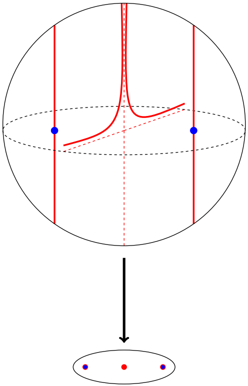
- If f defines an isolated singularity, then the normal cone to $C_{F\times\pi}$ is automatically flat over $\mathbb{C}^u$ — even at the origin!
- Any deformation through an ideal with respect to which f has finite extended codimension will automatically have flat normal cone away from the origin.
- Our hypotheses are independent of the constancy of the Lê numbers at the origin.

Example: $F((x,y,z),s,t) = (x^2 + y^2 z - s)(x - t)$



deforms to



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Results of Siersma let us consider the homological contribution of each critical value individually, so we have:

$$\tilde{H}_i(\mathbb{F}_f) \cong \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \\ \mathbb{Z}^{\oplus 3} & i = 2 \\ 0 & i > 3 \end{cases}$$

Idea of the Proof

Thom's First Isotopy Lemma

Lemma (Thom)

A proper stratified submersion is a fiber bundle smoothly along each stratum.

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Example

Let M be a compact manifold with boundary and N a smooth manifold. Then $\phi : M \to N$ is a fiber bundle over its image so long as $\phi|_{\mathring{M}}$ and $\phi|_{\partial M}$ are both submersions.

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Example

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Therefore the challenge in proving Milnor fibration-type results is to show that the smooth fibers meet the boundary sphere S_ε transversally.

The First Condition

- Our condition "For every critical point $p \neq 0$ of f near the origin, there exists a $V \in \Theta_{\mathbb{C}^{n+1}\times 0,p}$ such that $Vf = 0$ and $\left(\left(\sum_{i=0}^{n} \bar{x}_i dx_i\right) \cdot V\right)(p) \neq 0$." essentially means that there are holomorphic vector field germs witnessing the existence of f's Milnor fibration.

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- Our condition "For every critical point $p \neq 0$ of f near the origin, there exists a $V \in \Theta_{\mathbb{C}^{n+1} \times 0, p}$ such that $Vf = 0$ and $\left(\left(\sum_{i=0}^{n} \bar{x}_i dx_i\right) \cdot V\right)(p) \neq 0$." essentially means that there are holomorphic vector field germs witnessing the existence of f's Milnor fibration.
- It is then enough to note that the flatness of $C_{F \times \pi}$ over $\mathbb{C}^u$ is equivalent to the requirement that each vector field germ in $\ker Df$ can be extended to one in $\ker D(F \times \pi)$.

Thom Stratifications

- One way to prove the Milnor fibration for f exists: Stratify the ambient space in a way that imposes some control on the limits of tangent planes to smooth fibers at critical points. Such a stratification is said to satisfy the **Thom (a_f) condition**.

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- We can get this kind of control of limiting tangent planes by considering the **relative conormal space**

$$T^*_f\mathbb{C}^{n+1} := \overline{\{(x,\eta) \in T^*\mathbb{C}^{n+1} \mid x \notin C_f, \eta(T_x f^{-1}(f(x))) = 0\}}.$$

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- (Briançon-Maisonobe-Merle '94) We can get a Thom stratification using the fact that $T_f^*\mathbb{C}^{n+1}|_{V(f)}$ is the characteristic variety of a holonomic D-module.

The Second Condition

- The infinitesimal neighborhoods of $C_{F\times\pi}$ control the failure of the formation of the relative conormal $T^*_{F\times\pi}(\mathbb{C}^{n+1}\times\mathbb{C}^u)$ to commute well with restriction to a fiber of π.

The Second Condition

- The infinitesimal neighborhoods of $C_{F\times\pi}$ control the failure of the formation of the relative conormal $T^*_{F\times\pi}(\mathbb{C}^{n+1}\times\mathbb{C}^u)$ to commute well with restriction to a fiber of π.
- In particular, if the normal cone to $C_{F\times\pi}$ is flat, we can actually produce a stratification satisfying the Thom $(a_{F\times\pi})$ condition. This proves our result.

Bringing It Full Circle: Homogeneous Families

Some Simplifications

Suppose our map

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- Then, to apply our theorem, we can consider the *projective* vanishing $\Sigma_{F\times\pi}$ of $J_{F\times\pi}$ in $\mathbb{P}^n \times \mathbb{C}^u$ and verify only that $C_{\Sigma_{F\times\pi}}(\mathbb{P}^n \times \mathbb{C}^u)$ is flat over $\mathbb{C}^u$.

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- Moreover, the kind of splitting we saw earlier cannot occur — we are in fact guaranteed that, when the theorem applies, the Milnor fibration at the origin will remain constant.

Partitioning the Space of Hypersurfaces

Corollary (H '23)

Let $H_{n,d} \cong \mathbb{P}^{\binom{n+d}{n}-1}$ be the space of degree-d hypersurfaces in $\mathbb{P}^n$. Then iteratively taking the non-flat loci of the appropriate normal cones gives us a partition of $H_{n,d}$ into finitely many disjoint Zariski-locally-closed subsets so that the fiber diffeomorphism type of the affine Milnor fibrations of the corresponding defining polynomials is constant along each subset.

Thanks for listening!