Milnor Fiber Consistency via Flatness

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Introduction

The Milnor Fibration

Definition (Milnor '68)

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ. Its restriction

$$f : B_\varepsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

for $1 \gg \varepsilon \gg \delta > 0$ is a smooth locally trivial fibration over $D^*_\delta := D_\delta \setminus 0$, called the **Milnor fibration** *of f at the origin. Denote its fiber by $\mathbb{F}_f$.*

Milnor's Bouquet Theorem

Theorem (Milnor '68)

Suppose $\dim_0 \operatorname{Crit}(f) = 0$. *Then* $\mathbb{F}_f \simeq \bigvee_{\mu_f} S^n$, *where* $\mu_f := \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1},0}/J_f$ *is the* **Milnor number** *and*

$$J_f := \left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

is the **Jacobian ideal** *of* f.

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Thus, if we identify $\mathrm{Crit}(f)$ with $\mathrm{Spec}\, \mathcal{O}_{\mathbb{C}^{n+1},0}/J_f$, we can see that the Milnor fiber is determined by $\mathrm{Crit}(f)$. **Is this true in the non-isolated case as well?**

Algebraic Background

Normal Cone: Definition

It will turn out that we need to retain some information about the embedding of $\mathrm{Crit}(f)$ as well, using the following construction:

Definition

Let $X \hookrightarrow Y$ be a closed inclusion of schemes (or complex-analytic spaces) with ideal sheaf $\mathcal{I}$. Then the **normal cone** *$C_X Y$ of X in Y is the scheme (resp. $\mathbb{C}$-analytic space) given by the relative spectrum (resp. relative analytic spectrum) of the* **associated graded sheaf of algebras**

$$\mathrm{gr}_{\mathcal{I}} \mathcal{O}_Y := \bigoplus_{i=0}^{\infty} \mathcal{I}^i / \mathcal{I}^{i+1}.$$

Normal Cone: Intuition

- If X is a point, then $C_X Y$ is the **tangent cone** to Y at X.
- If X and Y are smooth, then $C_X Y$ is the **normal bundle** to X in Y.
- $X \hookrightarrow Y$ is a regular embedding if and only if $C_X Y$ is a vector bundle over X.

Flatness

We will prove results about the consistency of smooth fibers in a family (in some sense) following from the consistency of the critical loci. For the latter notion, we need:

Definition

Let $\phi : X \to Y$ be a morphism of schemes (or $\mathbb{C}$-analytic spaces). We say that ϕ is **flat** *if, for each point $p \in X$, $- \otimes_{\mathcal{O}_{Y,\phi(p)}} \mathcal{O}_{X,p}$ is an exact functor.*

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Geometrically, flatness is given in the Noetherian setting by the triviality of $C_{(\phi^{-1}(\phi(p)),p)}(X, p)$ over $C_{\phi(p)}(Y, \phi(p))$ at each point $p \in X$.

Results

Stratification Theorem

Theorem (H.)

Let X be a $\mathbb{C}$-analytic manifold, $\pi : X \times \mathbb{C}^u \to \mathbb{C}^u$ the projection, and $F : X \times \mathbb{C}^u \to \mathbb{C}$ a holomorphic function nowhere constant on fibers of π. Let $\mathrm{Crit}(F \times \pi)$ be the vanishing locus of the ideal sheaf of maximal minors of the Jacobian matrix of $F \times \pi$. Then there exists a $\mathbb{C}$-analytic Whitney stratification of $X \times \mathbb{C}^u$ so that:

- $(X \times \mathbb{C}^u) \setminus \mathrm{Crit}(F \times \pi)$ *is the ambient stratum.*
- *The non-flat locus of $C_{\mathrm{Crit}(F \times \pi)}(X \times \mathbb{C}^u)$ over $\mathbb{C}^u$ is a union of strata.*
- *The Thom $(a_{F \times \pi})$ condition with respect to the ambient stratum is satisfied on any stratum not contained in this non-flat locus.*

Setup: Germs of Families

We consider the following circumstances:

- $F : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}, 0)$ holomorphic
- $\pi : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}^u, 0)$ the projection
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- $f := F|_{\pi^{-1}(0)}$ the function being deformed
- $J_{F\times\pi} := \left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}\right)$ the ideal of maximal minors of the Jacobian matrix of $F \times \pi$
- $\mathrm{Crit}(F \times \pi) := \operatorname{Spec} \mathcal{O}_{\mathbb{C}^{n+1}\times\mathbb{C}^u,0\times 0}/J_{F\times\pi}$ the family of critical loci

Goal: Smooth Fiber Consistency

We wish to find circumstances in which the following is true:
For $1 \gg \varepsilon \gg \delta, \gamma > 0$, if we take representatives on $B_\varepsilon \times B_\gamma$ in $\mathbb{C}^{n+1} \times \mathbb{C}^u$ and let $\Delta^{\varepsilon}_{\delta,\gamma} := \overline{(F \times \pi)(\mathrm{Crit}(F \times \pi))} \cap (D_\delta \times B_\gamma)$ be the discriminant, the restriction

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap (F \times \pi)^{-1}((D_\delta \times B_\gamma) \setminus \Delta^{\varepsilon}_{\delta,\gamma}) \to (D_\delta \times B_\gamma) \setminus \Delta^{\varepsilon}_{\delta,\gamma}$$

is a diffeomorphically locally trivial fibration.

Theorem (H.)

Suppose that either of the following holds:

- *For every* $p \in \mathrm{Crit}(f)$ *in a sufficiently small punctured neighborhood of the origin, there exists a germ* V *of a holomorphic vector field at* p *such that* $Vf = 0$ *and* $V(p)$ *is not tangent to the sphere of radius* $|p|$ *centered at the origin. Moreover,* $\mathrm{Crit}(F \times \pi)$ *is flat over* $\mathbb{C}^u$ *everywhere on* $\pi^{-1}(0)$*, except possibly at the origin.*
- $C_{\mathrm{Crit}(F\times\pi)}(\mathbb{C}^{n+1} \times \mathbb{C}^u)$ *is flat over* $\mathbb{C}^u$ *everywhere on* $\pi^{-1}(0)$*, except possibly at the origin.*

Then we have smooth fiber consistency for $F \times \pi$ *in the sense previously discussed.*

Consequence: Homogeneous Polynomials

Corollary

Let $H_{n,d} \cong \mathbb{P}^{\binom{n+d}{n}-1}$ be the space of homogeneous degree-d polynomials in $n+1$ variables up to scaling. Then we can partition $H_{n,d}$ into finitely many Zariski-locally-closed subsets such that the diffeomorphism types of the polynomials' Milnor fibrations are constant along each subset.

Consequence: Homogeneous Polynomials

Proof idea

- *Let* $\Sigma_{n,d} \subset \mathbb{P}^n \times H_{n,d}$ *be the scheme whose fiber over each hypersurface is its singular locus.*
- *Find* $C_{\Sigma_{n,d}}(\mathbb{P}^n \times H_{n,d})$ *and let* $H'_{n,d} \subset H_{n,d}$ *be a closed subset such that the normal cone is flat over* $H_{n,d} \setminus H'_{n,d}$.
- *Pull everything back over* $H'_{n,d}$ *and repeat.*

$$\begin{tikzcd}
 & \cdots \arrow[r, hook] & \mathbb{P}^n \times H'_{n,d} \arrow[rr, hook] \arrow[dd] & & \mathbb{P}^n \times H_{n,d} \arrow[dd] \\
\cdots \arrow[r, hook] & \Sigma'_{n,d} \arrow[ur, hook] \arrow[dr] \arrow[rr, hook] & & \Sigma_{n,d} \arrow[ur, hook] \arrow[dr] & \\
 & \cdots \arrow[r, hook] & H'_{n,d} \arrow[rr, hook] & & H_{n,d}
\end{tikzcd}$$

Consequence: Critical Locus a Complete Intersection

Corollary

In the setting of the theorem, suppose we have a regular sequence $g_1, \ldots, g_c$ *generating an ideal* I *such that* $J_f \subseteq I$ *and* $\dim_{\mathbb{C}} I/J_f < \infty$. *Then we have smooth fiber consistency for* $F \times \pi$ *so long as there are deformations* G_i *of the* g_i *over* $\mathbb{C}^u$ *such that* $J_{F \times \pi} \subseteq (G_1, \ldots, G_c)$.

Thanks for listening!