Milnor Fiber Consistency via Flatness

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Abstract

The Milnor fibration gives a well-defined notion of the **smooth local fiber** of a holomorphic function at a critical point. Milnor's work in the isolated case suggests that this fiber's topology should be controlled by the scheme-theoretic invariants of the critical locus; we give results which demonstrate that this is true in a relative sense. Specifically, we show that the local smooth fiber varies nicely in families where the embedded critical locus satisfies certain algebraic consistency requirements and discuss implications for homogeneous polynomials and other special cases.

Background and Motivation

Milnor proved in [Mil68] that, if f defines an isolated singularity at the origin, then its **Milnor fiber** $\mathbb{F}_f$ is homotopy equivalent to $\bigvee_{\mu_f} S^n$, where

$$\mu_f := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{\left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}\right)}.$$

This is to say that the homotopy type of the Milnor fiber is determined entirely by the critical locus of f, *provided* we endow it with the nonreduced structure given by the **Jacobian ideal** $J_f := \left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}\right)$. This naturally leads us to ask:

Question What does the critical locus of f (with the non-reduced structure given by J_f) tell us about $\mathbb{F}_f$ in the non-isolated case?

Main Result

Theorem (H.) Let $F : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}, 0)$ be a germ of a family of holomorphic functions, where $\pi : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}^u, 0)$ is the projection onto the last u coordinates, which we regard as parameters. Suppose $f := F|_{\mathbb{C}^{n+1}\times 0}$ is non-constant and denote by $C_{F\times\pi}$ the union of the critical loci of the functions in the family, defined as the vanishing of the ideal $J_{F\times\pi} := \left(\frac{\partial F}{\partial x_0}, \cdots, \frac{\partial F}{\partial x_n}\right)$.

Suppose that the **normal cone** of $C_{F\times\pi}$ in $\mathbb{C}^{n+1+u}$ is flat over $\mathbb{C}^u$ everywhere on $\mathbb{C}^{n+1} \times 0$, except possibly at the origin. Then

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap F^{-1}(D_\delta) \to D_\delta \times B_\gamma$$

is a fibration away from the discriminant for $1 \gg \varepsilon \gg \delta, \gamma > 0$.

This tells us that, if the **scheme-theoretic critical loci of the functions vary consistently** in an **embedded sense**, the **smooth fibers vary consistently** as well.

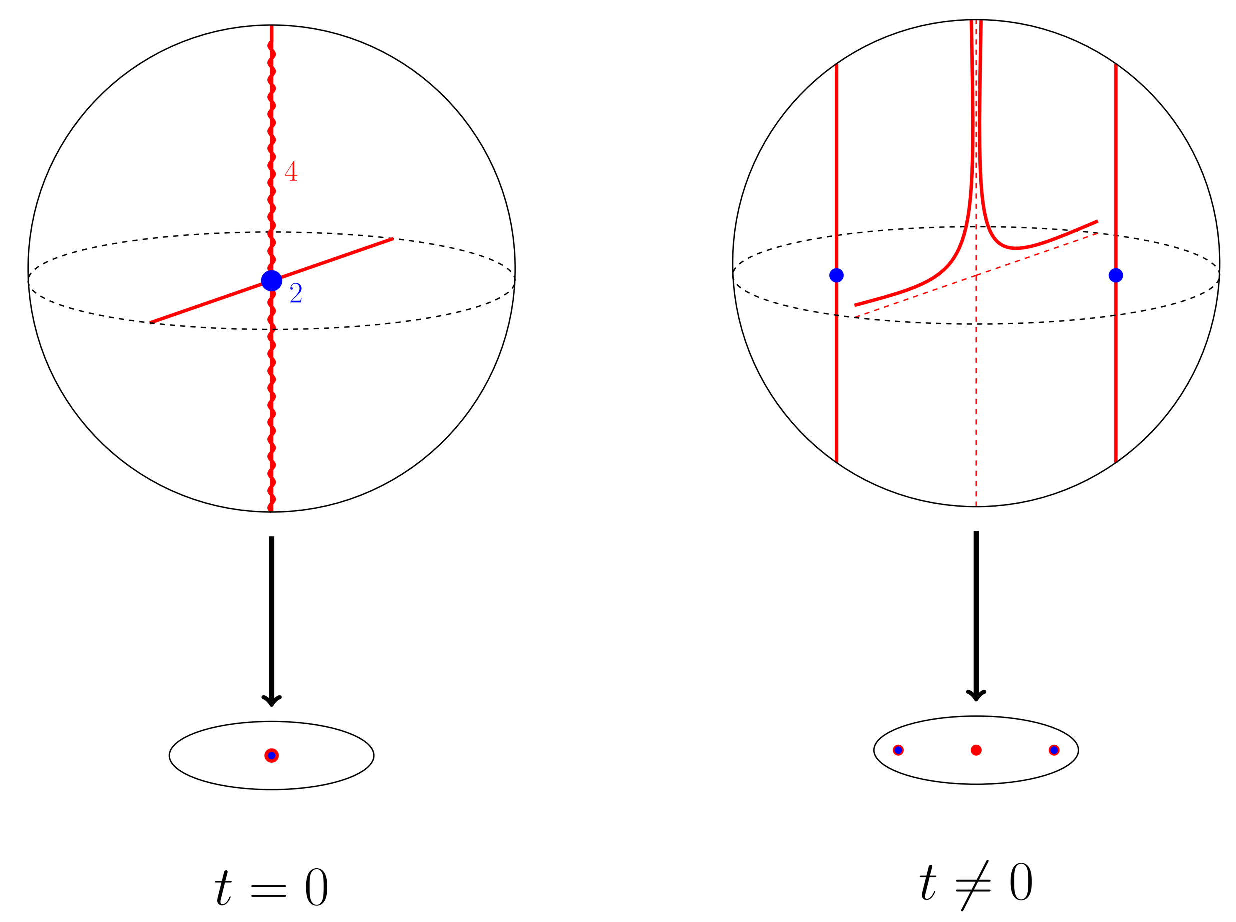
References Cited

[Mil68] John Milnor. *Singular Points of Complex Hypersurfaces*. Princeton, New Jersey: Princeton University Press and the University of Tokyo Press, 1968.

[ST17] Dirk Siersma and Mihai Tibăr. "Milnor Fibre Homology via Deformation". In: *Singularities and Computer Algebra*. Ed. by Wolfram Decker, Gerhard Pfister, and Mathias Schulze. Cham, Switzerland: Springer International Publishing, 2017, pp. 305–322. DOI: 10.1007/978-3-319-28829-1_14.

An Example Deformation

Let $f(x, y, z) = x^3 + xy^2z$. Then $F(x, y, z, t) = (x^2 + y^2z - 5t^2)(x - t)$ deforms the normal cone to the union of critical loci flatly, splitting off two D_∞ singularities. The critical loci can be depicted as follows:



By handling each connected component separately in the style of [ST17], we can then find that $\tilde{H}_1(\mathbb{F}_f) \cong \mathbb{Z}$, $\tilde{H}_2(\mathbb{F}_f) \cong \mathbb{Z}^{\oplus 3}$, and all other homology groups are zero.

Homogeneous Polynomials

For homogeneous polynomials, the situation is particularly nice, and as a result we can **partition the space of degree-d hypersurfaces** by the types of the corresponding Milnor fibrations:

Corollary (H.) Let $H_{n,d} \cong \mathbb{P}^{\binom{n+d}{n}-1}$ be the space of degree-d hypersurfaces in $\mathbb{P}^n$, so that the $\binom{n+d}{n}$ projective coordinates of each point give the coefficients up to scaling of the monomial terms in a homogeneous polynomial defining the corresponding hypersurface. Then iteratively applying the main result gives us a partition of $H_{n,d}$ into finitely many disjoint Zariski-locally-closed subsets so that the diffeomorphism types of the (affine) Milnor fibers of the polynomials corresponding to the points of $H_{n,d}$ are constant along each stratum, as are the monodromy diffeomorphisms.

Critical Locus a Complete Intersection

The main result also guarantees the well-behavedness of nice deformations of functions with critical locus a complete intersection:

Corollary (H.) Let $I = (g_1, \ldots, g_c)$ be an ideal defining a germ at the origin of a complete intersection of codimension c in $\mathbb{C}^{n+1}$. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ such that $J_f \subseteq I$ and $j(f) := \dim_{\mathbb{C}} I/J_f$ is finite.

Then, if $\tilde{I} = (G_1, \ldots, G_c)$ is a deformation of I, any deformation F of f which is compatible with it in the sense that $J_{F\times\pi} \subseteq \tilde{I}$ preserves the smooth local fiber.

Thus, in any deformation which **respects the complete intersection structure** of the critical locus, the **smooth fibers vary consistently**.