Ideal-Theoretic Study of the Milnor Fibration

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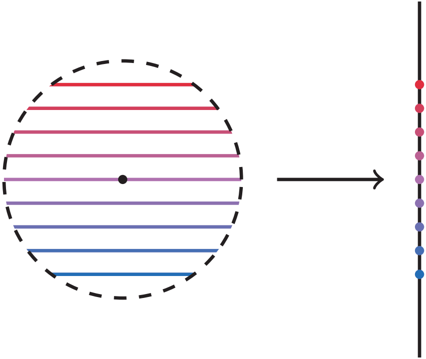
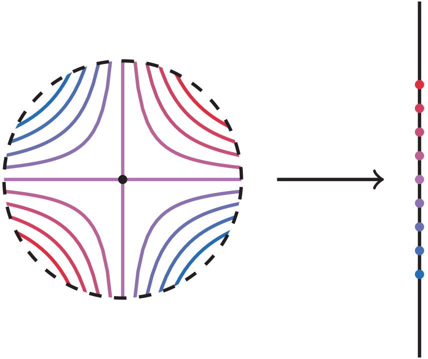
Non-Critical Points

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function. If f is a submersion at $p \in \mathbb{C}^{n+1}$, then f looks like a coordinate projection locally at p:



Critical Points

On the other hand, if f fails to be a submersion at $p \in \mathbb{C}^{n+1}$, the local fiber through p will be singular:



The Milnor Fibration

However, the smooth local fibers of f will still be consistent:

Theorem (Milnor '68, Lê '76)

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a holomorphic function germ. Its restriction*

$$f : B_\varepsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

for $1 \gg \varepsilon \gg \delta > 0$ *is a smooth locally trivial fibration over* $D_\delta^* := D_\delta \setminus 0$.

This is called the **Milnor fibration** of f at the origin — its fiber is denoted by $\mathbb{F}_f$.

Long-Term Goals

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Understand the behavior of the Milnor fibration of an arbitrary holomorphic function germ.

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Understand the behavior of the Milnor fibration of an arbitrary holomorphic function germ. Concretely, find effective means of computing the following:

- *The homology or homotopy type of the Milnor fiber.*
- *The monodromy of the Milnor fibration.*

First Relations with the Critical Locus

Theorem (Kato-Matsumoto '73)

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ. Let s denote the complex dimension of the critical locus of f at the origin. Then $\mathbb{F}_f$ is $(n-s-1)$-connected.

In particular, since $\mathbb{F}_f$ is a Stein manifold, $\tilde{H}_i(\mathbb{F}_f) = 0$ for $i \notin [n-s, n]$.

The Jacobian Criterion

We can also endow the critical locus with a non-reduced structure:

Theorem (Jacobian Criterion)

Let $F : X \to Y$ be a finitely-presented flat map of pure relative dimension n (e.g., $F : \mathbb{C}^{n+k} \to \mathbb{C}^k$ with n-dimensional fibers).

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The Milnor Number

This extra structure gives us information about the Milnor fibration:

Theorem (Milnor '68, Hamm '71)

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a non-constant holomorphic function germ such that* 0 *is an isolated point of* Σ_f. *Then* $\mathbb{F}_f \simeq \bigvee_{\mu_f} S^n$,

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$$\mu_f := \mathrm{length}(\mathcal{O}_{\Sigma_f,0}) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, \ldots, x_n\}}{\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right)}$$

is the **Milnor number***.*

Non-Isolated Critical Points

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Find an analogous way to compute data about the Milnor fibration from Σ_f (i.e., from J_f) in the case where 0 is a non-isolated critical point.

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For today, we will discuss a relative version of the preceding result for the non-isolated case.

Setup: Deformations

We will focus on 1-parameter deformations of holomorphic function germs. Consider:

- $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ — a non-constant holomorphic function germ
- $F : (\mathbb{C}^{n+2}, 0) \to (\mathbb{C}, 0)$ with $F(x_0, \ldots, x_n, t) = f_t(x_0, \ldots, x_n)$ such that $f_0 = f$ — a germ of a holomorphic deformation of f
- $\pi : (\mathbb{C}^{n+2}, 0) \to (\mathbb{C}, 0)$ with $\pi(x_0, \ldots, x_n, t) = t$ — the projection to the parameter space

Deformations Preserving the Local Smooth Fiber

In these circumstances, we know the following by definition for $1 \gg \varepsilon \gg \delta > 0$ and $|v| < \delta$:

$$f_0{}^{-1}(v) \cap B_\varepsilon \cong \mathbb{F}_f \text{ whenever } f_0{}^{-1}(v) \cap B_\varepsilon \text{ is smooth}$$

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We now want to know the circumstances under which the following is true for $1 \gg \varepsilon \gg \delta, \gamma > 0$, $|v| < \delta$, and $|t| < \gamma$:

$$f_t{}^{-1}(v) \cap B_\varepsilon \cong \mathbb{F}_f \text{ whenever } f_t{}^{-1}(v) \cap B_\varepsilon \text{ is smooth} \qquad (*)$$

That is, we want to know when the deformation can be used to study the Milnor fibration of f.

Counterexample

We can see that the condition $(*)$ is not always satisfied:

Example

Let $F(x, y, z, t) = xy - tz$. Then the condition $(*)$ does not hold — for example, ${f_t}^{-1}(0) \cap B_\varepsilon = \{z = \frac{xy}{t}\} \cap B_\varepsilon$ is smooth and contractible for arbitrarily small nonzero ε and t, but the Milnor fiber of $f = xy$ is homotopy-equivalent to S^1.

Ideal-Theoretic Control of Deformations

Theorem (H.)

Let f, F, and π be as before. Suppose that, for all $k \geq 0$, the kth-order infinitesimal neighborhood $V({J_{F\times\pi}}^{k+1})$ of $\Sigma_{F\times\pi}$ in $\mathbb{C}^{n+2}$ is flat over the parameter space $\mathbb{C}$ under the projection π.
(Equivalently: The normal cone to $\Sigma_{F\times\pi}$ in $\mathbb{C}^{n+2}$ is flat over the parameter space.)
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Then the condition $(*)$ holds.

That is, the condition $(*)$ holds as long as t is a non-zerodivisor in $\mathbb{C}\{x_0, \ldots, x_n, t\}/\left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}\right)^{k+1}$ for all $k \geq 0$. (Equivalently: In $\mathrm{gr}_{\left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}\right)} \mathbb{C}\{x_0, \ldots, x_n, t\}$.)

Notes

- $\mathrm{gr}_I R := R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$ is the **associated graded algebra** of I in R; its spectrum is the **normal cone** to $V(I)$ in $\operatorname{Spec} R$.
- It is possible to relax the hypothesis slightly in various ways, at the cost of making the statement more complicated.
- The conclusion is also stronger than stated; really we get a Milnor-Lê fibration for $F \times \pi$, whose existence implies the condition $(*)$ (and also allows us to retrieve monodromy information).

Idea of Proof (for the experts)

- We seek to apply Thom's first isotopy lemma by showing the transversality of smooth fibers of $F \times \pi$ to the boundary sphere family $S_\varepsilon \times \mathbb{C}$.
- Failures of transversality can be ruled out by producing a stratification (partially) satisfying the Thom condition.

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- Failures of transversality can be ruled out by producing a stratification (partially) satisfying the Thom condition.
- The Thom condition can be phrased in terms of the relative conormal space $T^*_{F\times\pi}\mathbb{C}^{n+2}$.
- The behavior of the relative conormal space under specialization to a fiber of π is controlled by the failure of flatness of the infinitesimal neighborhoods of $\Sigma_{F\times\pi}$ over the parameter space.

Thanks for listening!

Happy birthday, Pepe!