The Geometry of Rings and Schemes
Lecture 1: Introduction

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Classically, algebraic geometry studies solution sets to systems of polynomial equations — for example, given three circles in the plane, how many ways are there to draw a fourth circle tangent to all three? (This problem is often called *Apollonius' problem*, and valid choices of a fourth circle the *circles of Apollonius*.) These kinds of questions have been studied for centuries using various techniques and approaches — we will be interested in the language of *schemes*, which was developed by Grothendieck et al. in the mid-1900s and has come to be the framework of choice in the vast majority of modern work on the subject. Scheme theory provides a powerful and beautiful perspective which unifies classical questions of geometry with ideas from number theory, differential geometry, and other areas, and, as mentioned, developing some level of fluency in this way of thinking has become more or less essential for working algebraic geometers.

However, the theory is also notoriously difficult to learn — from anecdotal evidence, I have the impression that the median number of times a student takes an "introduction to schemes" course or undertakes an independent review of the same material before becoming comfortable with it is quite a bit higher than 1. My hope for this course is to skip some of this process by focusing on motivations and intuition for the core constructions and ideas, perhaps at the expense of not moving quite as quickly or covering as much ground as we might otherwise — ideally, we should build up the conceptual framework you need to approach the definitions we don't get around to covering on your own and understand them the first time through.

1 The Basic Idea

We will begin with two motivating propositions, the first on topological spaces and the second on rings. (Here, and throughout the rest of this course, a "ring" will mean "a commutative ring with an identity element" and a "ring map" will be a map of such rings — in particular, one taking identity element to identity element.)

Proposition 1. *Let X, Y be topological spaces and $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. For convenience, write $U_{\alpha\beta} := U_\alpha \cap U_\beta$ for $\alpha, \beta \in A$. Suppose we are given continuous maps $\phi_\alpha : U_\alpha \to Y$ for all $\alpha \in A$ such that, for all $\alpha, \beta \in A$, $\phi_\alpha|_{U_{\alpha\beta}} = \phi_\beta|_{U_{\alpha\beta}}$.*

Then there exists a unique continuous map $\phi : X \to Y$ such that, for every $\alpha \in A$, $\phi|_{U_\alpha} = \phi_\alpha$.

This is to say that functions on topological spaces can be defined and determined *locally* — that is, we can zoom in at each point and handle things up close without losing the larger picture. This fact also holds true, mutatis mutandis, for many kinds of objects we typically consider to be somehow "geometric" — for example, if we replace our topological spaces by smooth manifolds and our continuous maps by smooth maps — to the extent that it would not be unreasonable to take it as a basic foundational aspect of what it means for a certain kind of object to "have geometry".

Now let's look at an interesting fact about rings:

Proposition 2. *Let R, S be rings and $\{f_\alpha\}_{\alpha \in A}$ a collection of elements of R such that $(f_\alpha \mid \alpha \in A)$ is the unit ideal (i.e., the ideal (1) consisting of the whole ring). Recall that we have a canonical identification of the localizations $(R_{f_\alpha})_{f_\beta}$ and $(R_{f_\beta})_{f_\alpha}$ with the localization $R_{f_\alpha f_\beta}$ for $\alpha, \beta \in A$. Suppose we are given ring maps $\phi_\alpha : S \to R_{f_\alpha}$ for all $\alpha \in A$ such that, for all $\alpha, \beta \in A$, the compositions $S \xrightarrow{\phi_\alpha} R_{f_\alpha} \to R_{f_\alpha f_\beta}$ and $S \xrightarrow{\phi_\beta} R_{f_\beta} \to R_{f_\alpha f_\beta}$ of ϕ_α and ϕ_β with the natural localization maps are the same.*

Then there exists a unique ring map $\phi : S \to R$ such that, for every $\alpha \in A$, the composition $S \xrightarrow{\phi} R \to R_{f_\alpha}$ of ϕ with the natural localization map is ϕ_α.

As long as we ignore the fact that the maps are going in the wrong direction, this looks suspiciously similar to the first proposition! (Note that restricting a function to an open subspace is the same as composing it with that subspace's inclusion map — we'll see why $(f_\alpha \mid \alpha \in A)$ being the unit ideal is analogous to having an open cover a bit later on.) This suggests that there should somehow be an analogy between rings and geometric objects like smooth manifolds and topological spaces where localizations at single elements play the role of inclusions of open subspaces — our goal going forward will be to develop this analogy in more detail, which will lead us to the notion of a scheme.

Before we do so, a general note: The analogy we are starting to build here entails that most concepts in algebraic geometry will come with (at least) two interpretations, an algebraic one and a geometric one, as we see here with localizations at single elements. For anything algebro-geometric you learn, you should try to understand both sides of the picture — sometimes, this will entail spending some amount of time reconstructing one or the other, depending on the priorities of the source you're learning from, but the payoff is usually worth it.

Indeed, this way of looking at things provides a rubric for understanding the motivations for and origins of the various ideas and definitions one encounters in algebraic geometry. That is, some concepts will come from commutative algebra and number theory, and in this case we promote them to algebro-geometric concepts by finding the corresponding geometric interpretation; on the other hand, some concepts will come from topology or differential geometry, and we can generally make these algebro-geometric by reformulating them in categorical or algebraic (i.e., sheaf-theoretic) terms. Determining which of these processes a particular definition arises from will generally simplify the task of understanding it.

In terms of how we actually work with algebro-geometric concepts, the general idea is this: Have intuitions geometrically, and compute and prove things algebraically. The ability to take this perspective is, to my mind, the great selling point of algebraic geometry — it is an area of study which lets us combine the symbolic concreteness of algebra with the visual intuitiveness of geometry.

2 Closed Subspaces

We now resume our project of developing the analogy between rings and spaces — our next goal will be to make a case, albeit a bit of an informal one, as to what the analogue of a closed subspace should be. We'll begin with a simple example of taking a topological concept and reformulating it categorically.

Question. *Which ring should be analogous to the empty topological space $\emptyset$?*

As noted, we proceed by figuring out a way to characterize the empty space that uses concepts we can already translate to the context of rings — in this case, we do so categorically, which is to say in terms of maps.

Proposition/Definition 1. *The* **empty space** *is the unique topological space $\emptyset$ such that, for every topological space X, there exists a unique continuous function $\emptyset \to X$.*

(For those who know some category theory — this is just to say that $\emptyset$ is the *initial object* in the category of topological spaces.)

Hence we can reformulate our question to make it more tractable:

Question (revised)**.** *Which ring Z has the property that every ring R admits a unique ring map $R \to Z$?*

Note that here we are keeping in mind the idea that maps between rings "go the wrong way" when compared to those between topological spaces. This new version of the question is easier to answer:

Answer. *$Z = 0$, the zero ring (i.e., the ring whose underlying set is $\{0\}$ with addition and multiplication inherited from the integers — note that, within this ring, 0 is technically the multiplicative identity, so we haven't left the world of commutative rings with unity).*

For any R, the map $R \to 0$ will of course send every element to zero. We note that this answer fits nicely into our existing framework:

Remark 1. *For any topological space X, the unique map $\emptyset \hookrightarrow X$ is an open inclusion. Likewise, for any ring R, we can see that the unique map $R \to 0$ is a localization at a single element, since $0 = R_0$, the localization at the zero element of R.*

Now we make our informal case for the identity of the closed subspaces. In topology, we can characterize these as the complements of the open subspaces — in particular, for any open subspace $U \hookrightarrow X$, we have an obvious way to cook up a closed subspace $Z \hookrightarrow X$ such that $U \cap Z = \emptyset$.

Consider a ring R and pick an element $f \in R$. Then, as we've seen, we have the localization map $R \to R_f$, but we can also consider the quotient $R \to R/(f)$. Since quotients commute with localizations, we can see in particular that $R_f/(f) \cong (R/(f))_f$ — indeed, in this case, both of these rings are 0.

$$\begin{array}{ccc} R & \longrightarrow & R_f \\ \downarrow & & \downarrow \\ R/(f) & \longrightarrow & 0 \end{array}$$

So, if we do both the quotient and the localization, we get back our "empty space" ring 0, and this seems somewhat similar to the idea of finding a natural closed subspace intersecting with a given open subspace to the empty space. This leads us to the following:

Idea. *For R a ring, quotient maps $R \to R/I$ (with I an ideal) will play the role of closed inclusions into R.*

As with the open inclusions, this coheres with the idea that the empty space should be closed, since $R/(1) = 0$ for any ring R. This also allows us to make sense of the ideal-based condition in Proposition 2:

Remark 2. *If R is a ring and we have a collection $\{R \to R_{f_\alpha} \mid \alpha \in A\}$ of "open inclusions", we can now see what it should mean for this to be an "open cover". Thinking of the "closed inclusion" $R \to R/(f_\alpha)$ as the "complement" to $R \to R_{f_\alpha}$ for each $\alpha \in A$, we can regard the quotient $R \to R/(f_\alpha \mid \alpha \in A)$ as the "intersection of the complements of all our open subspaces", since it is the result of quotienting out by all of these elements at once. To ask that $(f_\alpha \mid \alpha \in A)$ be the unit ideal is precisely to ask that this intersection of complements be empty.*

3 Points

Our goal thus far has been to develop a geometric intuition for rings, but at this point we may still be unsatisfied — we've further developed our analogy with topological spaces, but it is still rather abstract and may not provide guidance on how to actually visualize any particular ring. To address this problem, we now seek an analogy for something very basic and foundational — the idea of a *point* in a topological space.

Again, our first pass making this notion portable will be to phrase it categorically, which is to say in terms of maps:

Proposition/Definition 2. *Let X be a topological space. A* **point** *of X is a continuous map $P \to X$, where $P = \{*\}$ the unique topological space whose underlying set has cardinality 1.*

We can thus begin by asking:

Question. *Which ring(s) should correspond to the one-point space P?*

There are multiple possible ways of characterizing P, suggesting different answers to this question. For our purposes, the one that turns out to be fruitful will be as follows:

Proposition/Definition 3. *The* **one-point space** *is the unique non-empty topological space which has no nontrivial subspaces (i.e., no subspaces other than itself and $\emptyset$).*

Now, although we've developed notions of "open subspaces" and "closed subspaces" for rings, I haven't told you yet what a "subspace" of a ring should be in general, and we probably won't commit to any particular formulation of this idea. That said, it's pretty clear that, whatever notion we might settle on, it should include our open and closed subspaces, so let's make due with them for now:

Question (revised)**.** *Which nonzero rings have no nontrivial quotients or localizations?*

This more precise formulation gives an easy path to the answer:

Answer. *Fields.*

Indeed, even requiring that a ring have no nontrivial quotients will force it to be a field and hence also have no non-trivial localizations, since we can consider the ideal generated by any non-zero non-invertible element. Requiring that a ring have no nontrivial localizations, on the other hand, turns out not to be enough due to the possibility of nilpotents — for example, Artinian local rings such as $\mathbb{C}[x]/(x^2)$ may satisfy this property while still admitting nontrivial quotients.

For the categorically-minded, we remark briefly on a possible alternate characterization of the one-point space P:

Remark 3. *By analogy to our approach for $\emptyset$, we could note that P is the final object in the category of topological spaces — that is, every topological space X admits a unique continuous map $X \to P$, taking everything to the single point. It then turns out that the analogous ring — i.e., the initial object in the category of rings — is the ring of integers $\mathbb{Z}$, leading us to suspect that a point of a ring R should be a map $R \to \mathbb{Z}$.*

However, it turns out that this doesn't give us a particularly fruitful theory, since there are far too many interesting rings which don't admit maps to $\mathbb{Z}$ — for example, there does not exist a map $k \to \mathbb{Z}$ for any field k, and hence no ring containing a field would have a point under this definition either. The idea that the category of rings somehow lacks the "correct" initial object — typically called the "field with one element" — has led to some interesting research into what is termed "$\mathbb{F}_1$-geometry", but this is beyond the scope of this course.

Anyway, our chosen characterization results in an important *disanalogy* between rings and topological spaces: While any point in the topological world, in and of itself, looks the same as any other, the same is not true for the world of rings, where it seems that points will come in many flavors. Indeed, our various analogues to the one-point space admit many interesting interrelations in their own right — otherwise, no one would need to study field theory.

We can now take our first stab at what it should mean to have a "point" of a ring:

Idea (questionable)**.** *For a ring R, ring maps $R \to k$ with k a field should play the role of inclusions of points into R.*

However, there is a huge problem with this approach, at least for our purposes: Field extensions exist! In particular, if we have a "point" $R \to k$, and $k \hookrightarrow k'$ is a field extension, the composition will give another "point" $R \to k'$. Hence, under this definition, the collection of points of any ring admitting at least one point would be a proper class — this would make the project of starting to visualize a ring by finding all its points somewhat difficult. (Despite this, there are contexts later on in algebraic geometry where this perspective is advantageous, as in the case of, e.g., *geometric points* of schemes defined over non-algebraically-closed fields.)

For now, we will resolve this problem by focusing on the "points" which are minimal — that is, those which do not arise from another map to a field by composition with a field extension. These are, incidentally, exactly those points which should naturally qualify as "subobjects" no matter our exact definition:

Idea (revised)**.** *For R a ring, the maps $R \to k$ for k field such that $R \to k$ is given by a quotient of a localization (arbitrary, not necessarily at just one element) will play the role of inclusions of points into R. (To be concrete: These will be the natural maps $R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ for prime ideals $\mathfrak{p} \subset R$.)*

It is a good exercise to verify the claims we've just made:

Exercise 1. *Let R be a ring, k a field, and $\phi : R \to k$ a ring map. Show that there exists a unique prime ideal $\mathfrak{p} \subset R$ and ring extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow k$ such that ϕ factors as the composition $R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow k$ of this ring extension with the natural map $R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.*

As always, it behooves us to check that our new addition to the analogy between rings and topological spaces is consistent with what came before. In this case, if we have a ring R and an element $f \in R$, then a point $R_f \to k$ naturally gives rise to a point $R \to R_f \to k$ since the composition of localizations is a localization. Likewise, for $I \subseteq R$ an ideal and $R/I \to k$ a point, the composition $R \to R/I \to k$ is a point of R by virtue of the facts that the composition of quotients is a quotient and that quotients and localizations commute. Hence the points of our open and closed subspaces are also points of the ambient space, exactly as in the topological case.

Exercise 2. *Let R be a ring, $f \in R$ an element, and $I \subseteq R$ an ideal. Show that the points of R which factor through $R \to R_f$ in this way are those corresponding to primes $\mathfrak{p} \not\ni f$, and that those which factor through $R \to R/I$ are those corresponding to primes $\mathfrak{p} \supseteq I$.*

Exercise 3. *Let R be a ring. Show that R has no points if and only if $R = 0$.*

4 The Spectrum of a Ring

Having figured out which ring maps are analogous to the inclusions of points, we are now ready to make our geometric analogy much more explicit — specifically, we will produce for any ring R the topological space "most closely approximating the geometry of R", whose points are exactly the points of R. As we will see by examples, however, some information will be lost — this space will not keep track of *all* of the geometry of R, which is why this is a course about schemes and not just topological spaces.

Definition 1. *Let R be a ring. Then the* **spectrum** $\operatorname{Spec} R$ *of R is a topological space defined as follows. The underlying set is*

$$\{\phi : R \to k \mid k \text{ is a field and } \phi \text{ is given by composing a quotient and a localization}\},$$

the set of points of R. (This can be identified with the set $\{\mathfrak{p} \subset R \mid \mathfrak{p}$ is a prime ideal$\}$, as previously noted, which is the traditional approach.) The topology, called the **Zariski topology**, *is given by either of the following equivalent characterizations:*

1. *The closed sets are exactly those of the form*

$$\{\phi \in \operatorname{Spec} R \mid \phi \textit{ factors through } R \to R/I\}$$

   for ideals $I \subseteq R$.

2. *The sets*

$$\{\phi \in \operatorname{Spec} R \mid \phi \textit{ factors through } R \to R_f\}$$

   for elements $f \in R$ form a base for the topology. (In particular, the open sets are given by arbitrary unions of sets of this form.)

Exercise 4. *Check that the two given definitions of the Zariski topology are well-formed and agree with each other. (Hint: For the first, and to verify that we actually have a valid base in the second, it may be useful to look back over the parts of this lecture where we mentioned "intersections" for inspiration.)*

Exercise 5. *Show that* Spec *is functorial — that is, for $\phi : R \to S$ a ring map, there is an induced map $\operatorname{Spec}\phi : \operatorname{Spec} S \to \operatorname{Spec} R$ (Note the change in direction!) such that $\operatorname{Spec}\mathrm{id}_R = \mathrm{id}_{\operatorname{Spec} R}$ and $\operatorname{Spec}(\psi \circ \phi) = (\operatorname{Spec}\phi) \circ (\operatorname{Spec}\psi)$.*

Exercise 6. *Find the spectra of the following rings: $\mathbb{Q}, \mathbb{C}, \mathbb{C}[x], \mathbb{C}[x]_x, \mathbb{C}[x]_{(x)}, \mathbb{Z}$, and $\mathbb{C}[x, y]$.*

(This last ring, $\mathbb{C}[x, y]$, is much more difficult than the others — for a hint, see Example 7 of Section 3.2 in Vakil and the exercise which follows it.)

Exercise 7 (after computing $\operatorname{Spec}\mathbb{C}[x]$)**.** *Find two non-isomorphic quotients of $\mathbb{C}[x]$ whose spectra give the same closed subspace of $\operatorname{Spec}\mathbb{C}[x]$.*

Exercise 8 (after computing $\operatorname{Spec}\mathbb{Z}$)**.** *Recall the notion of the characteristic of a field and give a geometric interpretation.*

Exercise 9. *Let R be a ring. Show that any open cover of $\operatorname{Spec} R$ admits a finite subcover.*

(This property is usually called being "compact" in topological circles — however, in algebraic geometry we call spaces satisfying it *quasicompact*, out of deference to the fact that it will not actually turn out to give the right algebro-geometric analogue to the role played by compactness in topology.)