The Geometry of Rings and Schemes
Lecture 3: Fiber Products and Such

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So far, we've seen that rings have some kind of geometric structure and used this to construct more general objects called schemes which are locally isomorphic to rings. However, there are still many things we know how to do with topological spaces but not with schemes — for example, taking fibers of a map. Our methodology for solving such issues so far has been to figure out how to reformulate each construction in terms which lend themselves well to generalization — to this end, we will now spend some time discussing an important construction for topological spaces which encapsulates several different notions we may be interested in defining for schemes.

1 Fiber Products of Topological Spaces

Definition 1. *Let X, Y, Z be topological spaces and $f : X \to Z$ and $g : y \to Z$ continuous maps. Then the* **fiber product** *of X and Y over Z is the topological space*

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

which comes with the natural projections to X and Y induced by those of $X \times Y$.

By construction, we can see that the compositions of f and g with these natural projections agree — that is, the following diagram commutes:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

To see why this construction is important, consider the following special cases:

Example 1. *If $Z = \{*\}$ in the one-point space, then $X \times_Z Y = X \times Y$ is the usual product of topological spaces.*

*First draft of the TeX source provided by Márton Beke.

Example 2. *If $g : Y \hookrightarrow Z$ is the inclusion of a subspace, then $X \times_Z Y = f^{-1}(Y)$ is this subspace's inverse image under f, with the natural projections to X and Y giving the inclusion $f^{-1}(Y) \hookrightarrow X$ and the restricted map $f|_{f^{-1}(Y)}$ respectively.*

Example 3. *As a special case of the previous example: If $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$ are both inclusions of subspaces, then $X \times_Z Y = X \cap Y$ is their intersection in Z.*

For those who know some differential geometry, we have:

Example 4. *If $g : Y \to Z$ is the projection of a vector bundle (or, indeed, any fiber bundle) over Z, then $X \times_Z Y \to X$ is the projection of the pullback bundle f^*Y.*

Exercise 1. *Verify the preceding examples.*

Example 4 hints at the motivation for the term "fiber product" — in general, we can see that the fiber of the projection $X \times_Z Y \to X$ over a point $x \in X$ is exactly the fiber $g^{-1}(f(x))$ of g over $f(x)$.

To work with fiber products more fluently, we introduce some additional terminology:

Definition 2. *We call a diagram of the form*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

(or any isomorphic diagram) a **pullback square**, *or* **fiber square**. *We call the map $g' : X \times_Z Y \to X$ the* **pullback of** *g* **along** *f, and use similar terminology for f' by symmetry.*

For a given property P of continuous maps, if a map g having P implies that g' has P for all such pullback squares, we say that P is **preserved under pullback**.

Many important properties of continuous maps turn out to be preserved under pullback. Example 2 implies:

Example 5. *Being an open inclusion is preserved under pullback, as is being a closed inclusion.*

Indeed, we often *define* continuous maps as maps of sets which make one or the other of the preceding statements true.

Likewise, Example 4 gives another instance of such a property:

Example 6. *Being (the projection of) a vector bundle is preserved under pullback; the same is true of fiber bundles in general.*

We now want to replicate our definitions in the scheme-theoretic context — as usual, we begin by reformulating things categorically, in terms of continuous maps:

Proposition 1. *$X \times_Z Y$ is determined by the universal property expressed in the following diagram:*

$$\begin{tikzcd}
\forall W \arrow[dr, dashed, "\exists!\chi"] \arrow[drr, bend left, "\forall\psi"] \arrow[ddr, bend right, "\forall\phi"'] & & \\
& X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"'] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

That is, for every choice of topological space W and continuous maps $\phi : W \to X$ and $\psi : W \to Y$ such that $f \circ \phi = g \circ \psi$, there exists a unique continuous map $\chi : W \to X \times_Z Y$ such that $\phi = g' \circ \chi$ and $\psi = f' \circ \chi$. This property determines $X \times_Z Y$ and its projections uniquely up to isomorphism.

This way of phrasing things can be used to prove an important fact — namely, that "being a pullback is preserved under pullback". The precise meaning of this is expressed in the following two exercises:

Exercise 2. *Let $\tilde{X}, X, Y, Z$ be topological spaces and fix continuous maps $\tilde{X} \to X$, $X \to Z$, $Y \to Z$:*

$$\begin{tikzcd}
& X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
\tilde{X} \arrow[r] & X \arrow[r] & Z
\end{tikzcd}$$

Show that $\tilde{X} \times_X (X \times_Z Y) \cong \tilde{X} \times_Z Y$.

Exercise 3. *Let $X, Y, Z, \tilde{Z}$ be topological spaces and fix continuous maps $X \to Z$, $Y \to Z$, $\tilde{Z} \to Z$:*

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] & \\
X \arrow[r] & Z & \tilde{Z} \arrow[l]
\end{tikzcd}$$

Show that $(X \times_Z Y) \times_Z \tilde{Z} \cong (X \times_Z \tilde{Z}) \times_{\tilde{Z}} (Y \times_Z \tilde{Z})$.

In particular, by pulling back along open inclusions, we can see that the fiber product of X and Y over Z can be constructed locally on X, Y, and Z and glued together over open covers. This prepares us to define...

2 Fiber Products of Schemes

Since we have a characterization of the fiber product of topological spaces in terms of its universal property, we can now ask if there is an equivalent construction in the world of schemes. However, as is often the case, it will turn out to be easier to first address this question for affine schemes specifically — that is, for rings — and then proceed by gluing things together along affine open covers. As such, we ask:

Question. *Let R, S, T be rings and $r : T \to R$, $s : T \to S$ ring maps. Does there exist a ring P together with ring maps $s' : R \to P$, $r' : S \to P$ satisfying the universal property expressed by the following diagram?*

$$\begin{tikzcd}
T \arrow[r, "s"] \arrow[d, "r"'] & S \arrow[d, "r'"'] \arrow[ddr, bend left, "\forall b"] & \\
R \arrow[r, "s'"] \arrow[drr, bend right, "\forall a"'] & P \arrow[dr, dashed, "\exists ! c"] & \\
& & \forall Q
\end{tikzcd}$$

That is, are there such P, r', and s' so that, for every choice of ring Q and ring maps $a : R \to Q$ and $b : S \to Q$ satisfying $a \circ r = b \circ s$, there exists a unique ring map $c : P \to Q$ satisfying $a = c \circ s'$ and $b = c \circ r'$?

Answer. *Yes; this is the* **tensor product** $R \otimes_T S$.

In the first lecture, we discussed the idea that algebro-geometric concepts can generally be seen as arising either algebraically or geometrically, the goal of the student being to fill in the other half of the picture in either case. In this instance, we have something we really could have approached from either angle, either by starting with the tensor product and attempting to tease some kind of geometric meaning out of it through analogy or by proceeding as we actually did. In either event, the result is illustrative of the benefits of the algebro-geometric way of thinking, even just for the purposes of commutative algebra — tensor products can be a bit mystifying when first encountered from the perspective of ring theory, but building up a geometric understanding of the fiber product will allow you to deal with them on a much more intuitive level.

We can now define fiber products for schemes:

Definition 3. *Let $\operatorname{Spec} R \to \operatorname{Spec} T$ and $\operatorname{Spec} S \to \operatorname{Spec} T$ be maps of affine schemes. We define the* **fiber product** *of $\operatorname{Spec} R$ and $\operatorname{Spec} S$ over $\operatorname{Spec} T$ by*

$$(\operatorname{Spec} R) \times_{\operatorname{Spec} T} (\operatorname{Spec} S) := \operatorname{Spec}(R \otimes_T S).$$

If $\phi : X \to Z$ and $\psi : Y \to Z$ are maps of arbitrary schemes, we define the **fiber product** *$X \times_Z Y$ of X and Y over Z affine-locally. That is, for an affine open cover $\{W_\gamma\}_{\gamma \in \Gamma}$ of Z and affine open covers $\{U_{\gamma,\alpha}\}_{\alpha \in A_\gamma}$ and $\{V_{\gamma,\beta}\}_{\beta \in B_\gamma}$ of $\phi^{-1}(W_\gamma)$ and $\psi^{-1}(W_\gamma)$ respectively for each $\gamma \in \Gamma$, we construct $X \times_Z Y$ by gluing together the products $U_{\gamma,\alpha} \times_{W_\gamma} V_{\gamma,\beta}$ compatibly with the open covers.*

Of course, our immediate concern is to verify that this is well-defined and satisfies the properties we expect of a fiber product:

Exercise 4. *Check that this gluing makes sense for a given open cover and that the resulting scheme satisfies the universal property of Proposition 1 (with all topological spaces and continuous maps replaced by schemes and maps of schemes respectively). Conclude that the scheme constructed is independent of the chosen affine open covers.*

We can now make precise several things which we were previously a bit cavalier about. For example, in our first lecture's motivating proposition on affine open covers of affine schemes (which we expressed at the time in the language of rings), we treated $\operatorname{Spec} R_{fg}$ as the intersection of $\operatorname{Spec} R_f$ and $\operatorname{Spec} R_g$ in $\operatorname{Spec} R$ on the basis that $R_{fg} \cong (R_f)_g \cong (R_g)_f$ is the "result of doing both localizations". Knowing now that fiber products (and thus, in particular, intersections) should be given by tensor products on the level of rings, we can verify the correctness of this decision by observing that $R_{fg} = R_f \otimes_R R_g$. Similarly, in motivating our view of quotient maps as corresponding to closed inclusions, we touched on the idea that the intersection of $\operatorname{Spec} R_f$ and $\operatorname{Spec} R/I$ in $\operatorname{Spec} R$ should be the spectrum of $R_f/IR_f \cong (R/I)_f$, the "result of doing both the localization and the quotient" — and, indeed, this is precisely $R_f \otimes_R R/I$. Finally, we can likewise see that the intersection of closed subschemes $\operatorname{Spec} R/I$ and $\operatorname{Spec} R/J$ in $\operatorname{Spec} R$ is exactly what we expect — that is, $(R/I)/J(R/I) \cong (R/J)/I(R/J) \cong R/(I+J)$ is simply $R/I \otimes_R R/J$.

The natural next step, of course, is to explore the definitions of notions like inverse images of subschemes, fibers over particular points, products of schemes, and so forth given to us by our fiber product. First, however, we will clarify what we mean by "subschemes" exactly, at least in the open and closed cases. In the case of open inclusions, which were central to our definition of schemes in the first place, the definition is correspondingly immediate:

Definition 4. *An* **open subscheme** *of a scheme $(X, \mathcal{O}_X)$ is simply an open subspace on the level of ringed spaces — that is, it consists of a pair $(U, \mathcal{O}_U)$ such that U is an open subspace of X and $\mathcal{O}_U := \mathcal{O}_X|_U$ is the restriction of the structure sheaf. The inclusion map of such a subscheme is given by $(i, i^\#)$ for $i : U \hookrightarrow X$ the inclusion and $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_U$ the map of sheaves given on each open subset V by the restriction map from V to $U \cap V$ in $\mathcal{O}_X$.*

For closed inclusions, we require a different approach, inspired by the following fact from topology:

Proposition 2. *Let $i : X \to Y$ be a map of topological spaces and $\{V_\alpha\}_{\alpha \in A}$ an open cover of Y. Then i is the inclusion of a closed subspace if and only if $X \times_Y V_\alpha \to V_\alpha$ is for all $\alpha \in A$.*

That is, being a closed inclusion is a local condition on the target. Hence, we can define:

Definition 5. *A* **closed inclusion of affine schemes** *is the map $\operatorname{Spec} R/I \to \operatorname{Spec} R$ induced by the quotient map $R \to R/I$ for some ring R and ideal $I \subseteq R$.*

A map $i : X \to Y$ of arbitrary schemes is said to be a **closed inclusion** *if it is a closed inclusion of affine schemes affine-locally on the target — that is, for some (equivalently, any) affine open cover $\{V_\alpha\}_{\alpha \in A}$ of Y, the maps $X \times_Y V_\alpha \to V_\alpha$ for $\alpha \in A$ are all closed inclusions of affine schemes.*

We also introduce the following definition:

Definition 6. *Let X be a scheme. A* **locally closed subscheme** *of X is any scheme which can be realized as the intersection of an open subscheme of X with a closed one — that is, one of the form $U \times_X Z$ for $U \hookrightarrow X$ an open inclusion and $Z \hookrightarrow X$ a closed inclusion, with inclusion into X given by the natural map $U \times_X Z \to X$.*

Of course, all of these concepts are preserved under pullback:

Proposition 3. *Let $i : Y \hookrightarrow X$ be an open inclusion, closed inclusion, or locally closed inclusion. Then, for any map $X' \to X$ of schemes, the pulled-back map $i' : Y \times_X X' \to X'$ is also an open inclusion, closed inclusion, or locally closed inclusion respectively.*

Now we can, for the sake of explicitness, write down definitions for some scheme-theoretic analogues of topological concepts:

Definition 7. *Let $\phi : X' \to X$ be a map of schemes. Suppose that $i : Y \hookrightarrow X$ is an open inclusion, closed inclusion, locally closed inclusion, or a* **point inclusion** *— that is, for some affine open $j : \operatorname{Spec} R \hookrightarrow X$ and prime ideal $\mathfrak{p} \subset R$, i is the composition of j with the map on spectra induced by the natural map $R \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then we define the* **inverse image** *of Y under ϕ by $\phi^{-1}(Y) := Y \times_X X'$. In the last case, where i is the inclusion of a point $y = Y$, we also call this the* **fiber** *of ϕ over y and may also denote it by X'_y.*

As mentioned, we can also hope that knowing what a fiber product is will allow us to define a notion of "the product of two schemes" without reference to a choice of a pair of maps to some third scheme. We saw in Example 1 that we can do this for topological spaces by applying the fiber product construction with the canonical maps from both factors to the one-point space $\{*\}$. Now, there does exist a scheme to which every scheme admits a unique map — because each ring R admits a unique map $\mathbb{Z} \to R$, we find that every affine scheme carries a unique map to $\operatorname{Spec} \mathbb{Z}$. Since maps of schemes are locally determined, we hence get a unique map from an arbitrary scheme to $\operatorname{Spec} \mathbb{Z}$ by considering the maps on any affine cover and gluing them together.

Hence, we *could* define "the product of two schemes X and Y" to be $X \times_{\operatorname{Spec} \mathbb{Z}} Y$, the fiber product over $\operatorname{Spec} \mathbb{Z}$ given by the unique maps $X \to \operatorname{Spec} \mathbb{Z}$ and $Y \to \operatorname{Spec} \mathbb{Z}$. However, as mentioned when we first discussed how to define "points" in the context of rings, $\operatorname{Spec} \mathbb{Z}$ is in many respects a poor analogue for the one-point space, and we would often like to consider instead the schemes we have already designated as "points" — that is, the spectra of fields.

The difficulty in taking this perspective for our purposes now, of course, is that schemes do not a priori come with distinguished maps to any particular field spectrum. To remove this issue, essentially by fiat, we work with...

3 Schemes Over a Scheme

To sound slightly less silly, we could also call these objects "relative schemes". The main idea, as noted, is to fix some particular scheme and try to treat it as a "final object" like $\{*\}$ or $\operatorname{Spec} \mathbb{Z}$ — that is, one to which everything has a unique map. In practice, we do this by simply fixing such maps:

Definition 8. *Let S be a scheme. A* **scheme over** *S, or S-***scheme***, is a scheme X together with a chosen map $X \to S$, called the* **structure map** *of X over S. If X and Y are schemes over S, a* **map of schemes over** *S from X to Y is a scheme map $X \to Y$ which composes with the structure map of Y to the structure map of X — that is, one such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

If $S = \operatorname{Spec} R$ for some ring R, we often drop "Spec" from the terminology and speak simply of "R-schemes" or "schemes over R". In the case where R is moreover a field, we refer to it as the **ground field**.

This last situation is most common — as mentioned, we typically want our "final object" to be the spectrum of some ground field k, by analogy to the one-point space in topology (or, indeed, differential geometry). To understand what working over k means algebraically, note that affine k-schemes are precisely the spectra of k-algebras — that is, of rings R with some chosen inclusion $k \hookrightarrow R$ — and the maps of affine schemes over k are precisely those induced by k-algebra maps on the level of rings. Hence k-schemes are exactly the objects we would have gotten if we worked from the beginning with k-algebras and k-algebra maps in place of rings and ring maps. (By the same token, or just by recalling that $\operatorname{Spec} \mathbb{Z}$ is already a "final object" for schemes, we can see that "$\mathbb{Z}$-schemes" and their maps are the same thing as schemes and their maps — there is no map of rings which is not a map of $\mathbb{Z}$-algebras.)

Exercise 5. *Observe that, since the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ induces a map $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, every $\mathbb{C}$-scheme can naturally be regarded as an $\mathbb{R}$-scheme. Give a pair of $\mathbb{C}$-schemes X, Y and a map of $\mathbb{R}$-schemes $X \to Y$ which is not a map of $\mathbb{C}$-schemes.*

(Hint: Work with affine schemes — what is this question asking if we phrase it in terms of algebras?)

As promised, if we work over a given scheme S, we have a natural concept of the "product" of two schemes — for S-schemes X and Y, this is $X \times_S Y$. (We typically retain S in the notation, writing $\times_S$ instead of just $\times$, to emphasize which setting we are working in.)

Exercise 6. *Let k be a field. Show that $\mathbb{A}^2_k \cong \mathbb{A}^1_k \times_k \mathbb{A}^1_k$. (Recall: Here $\times_k$ is shorthand for $\times_{\operatorname{Spec} k}$.)*

(More generally: Show for integers $n \geq 0$ that $\mathbb{A}^n_k \cong \underbrace{\mathbb{A}^1_k \times_k \cdots \times_k \mathbb{A}^1_k}_{n \text{ copies}}$.)

In particular, by our previous examinations of $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{A}^2_{\mathbb{C}}$, we can see that the underlying topological space of the fiber product is *not* in general given by the fiber product of the underlying topological spaces. However, the properties of the tensor product guarantee that, for k-schemes X and Y and a given field extension $k \hookrightarrow \ell$, the *set* of k-scheme maps $\operatorname{Spec} \ell \to X \times_k Y$ *will* be exactly the product of the sets of k-scheme maps $\operatorname{Spec} \ell \to X$ and $\operatorname{Spec} \ell \to Y$; the only issues are that not all of these maps are necessarily inclusions of points in our most stringent sense of corresponding to a prime ideal in an affine patch and that we have said nothing yet about the topology on this product of sets.

In the case where $\ell = k$, however, we find that a map of k-algebras to ℓ is necessarily surjective, and so any map of k-schemes $\operatorname{Spec} k \to X$ defines a closed point. In the case where $k = \bar{k}$ is an algebraically closed field, the Nullstellensatz tells us that all closed points of the affine spaces $\mathbb{A}^n_k$ (and hence of any locally closed subscheme of such a space) are of this form — in particular, for such schemes, $\times_k$ does induce the standard product on the level of sets of closed points. (Observe by returning to $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{A}^2_{\mathbb{C}}$, however, that the induced topology is not the product topology.)

Sometimes, we want to think of the fiber product as changing the scheme we are working over:

Definition 9. *Let $S' \to S$ be a map of schemes. Then the operation taking an S-scheme X to the S'-scheme $X \times_S S'$ is called* **base change** *from S to S'.*

As an example of the notion of base change in action, we observe for fixed $n \geq 0$ that all of our "different affine n-spaces" — that is, all $\mathbb{A}^n_k$ for different choices of a field k — arise from one object over $\mathbb{Z}$ via base change:

Exercise 7. *Let k be a field, $n \geq 0$ an integer, and $\mathbb{A}^n_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. Show that $\mathbb{A}^n_k \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} k$.*

Inspired by this, we can expand our notion of "affine n-space":

Definition 10. *Let S be a scheme and $n \geq 0$ an integer. We define* **affine n-space over** *S by $\mathbb{A}^n_S := \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S$ — observe that the natural projection of the fiber product to the factor S allows us to view this as a scheme over S.*

Those who have seen some differential geometry may find it helpful to view the map $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ as the "universal trivial rank-n vector bundle" from which all other trivial rank-n vector bundles arise via pullback — we have not yet defined what we mean by a "vector bundle" in the scheme context, but recall from our discussion last week that we regard $\mathbb{A}^n_k$ as "the geometric realization of the vector space k^n", so we can at least see the phenomenon on the level of fibers:

Exercise 8 (basically Exercise 7 again)**.** *Let S be a scheme, $n \geq 0$ an integer, and $\operatorname{Spec} k \hookrightarrow S$ the inclusion of a point. Verify that the fiber of $\mathbb{A}^n_S$ over this point is exactly $\mathbb{A}^n_k$.*

(Compare this to differential geometry, where the universal trivial rank-n vector bundle is simply $\mathbb{R}^n \to \{*\}$.)

Affine spaces over schemes can be used to define a very typical constraint we impose on schemes and their maps:

Definition 11. *A map $\phi : X \to Y$ of schemes is said to be of* **finite type** *if there exists an open cover $\{V_\alpha\}_{\alpha \in A}$ such that each $\phi^{-1}(V_\alpha)$ admits an open cover by finitely many subschemes $U_{\alpha,1}, \ldots, U_{\alpha,k_\alpha}$ so that each $\phi|_{U_{\alpha,i}} : U_{\alpha,i} \to V_\alpha$ factors as*

$$U_{\alpha,i} \hookrightarrow \mathbb{A}^{n_{\alpha,i}}_{V_\alpha} \to V_\alpha$$

for $U_{\alpha,i} \hookrightarrow \mathbb{A}^{n_{\alpha,i}}_{V_\alpha}$ a closed inclusion and $\mathbb{A}^{n_{\alpha,i}}_{V_\alpha} \to V_\alpha$ the natural map.

If we relax the definition by allowing our open covers of the $\phi^{-1}(V_\alpha)$ to be infinite, ϕ is instead said to be **locally of finite type***.*

For S a scheme, we define a **(locally) finite-type scheme over** *S to be a scheme over S such that the structure morphism is (locally) of finite type.*

To understand this definition, observe that we can take our open cover of Y to be affine without loss of generality, and that over an affine patch $V_\alpha = \operatorname{Spec} R$ each composition $U_{\alpha,i} \hookrightarrow \mathbb{A}^{n_{\alpha,i}}_{V_\alpha} \to V_\alpha$ is given on the level of rings by $R \to R[x_1, \ldots, x_{n_{\alpha,i}}] \to R[x_1, \ldots, x_{n_{\alpha,i}}]/I_{\alpha,i}$ for some ideal $I_{\alpha,i}$. That is, for one scheme to be locally of finite over another means essentially that it is affine-locally constructed out of finitely-generated algebras; requiring it to be of finite type (without the "locally") simply imposes an additional finiteness condition of a more topological nature.

In practice, a great deal of algebraic geometry happens in the setting of finite-type schemes over a field (particularly an algebraically closed one). In particular, the study of finite-type schemes over $\mathbb{C}$ is closely connected to complex algebraic geometry in the classical sense — from the definition, we can see that such an object is obtained by gluing together finitely many closed subschemes of affine spaces $\mathbb{A}^n_{\mathbb{C}}$, which is the scheme-theoretic way to say "finitely many spaces cut out in $\mathbb{C}^n$ by polynomial equations".

Example 7. *Let $X = \operatorname{Spec} \mathbb{C}[x]_x$. Because X includes into the affine line $\operatorname{Spec} \mathbb{C}[x]$ as an open subset, rather than a closed one, it may be tempting to think that X is not a finite-type $\mathbb{C}$-scheme. However, the isomorphism $\mathbb{C}[x]_x \cong \mathbb{C}[x, y]/(xy-1)$ of $\mathbb{C}$-algebras which identifies x with x and x^{-1} with y allows us to realize X as a closed subscheme of $\mathbb{A}^2_{\mathbb{C}}$, so it turns out to be of finite type over $\mathbb{C}$ after all. Hence the setting of finite-type schemes over a field is slightly less restrictive than it may appear at first.*