The Geometry of Rings and Schemes
Lecture 4: Some Properties of Schemes

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At this point we have discussed how to view rings as geometric objects, introduced the idea of schemes as the more general class of such objects which comes from allowing ourselves to glue rings together, and developed the scheme-theoretic analogues to several fundamental topological concepts. We will now turn our attention to some features more particular to the algebro-geometric setting, which are uninteresting or ill-defined for at least the types of topological spaces we are most used to working with outside of algebraic geometry.

1 Local Rings

We begin by introducing a core component in the scheme-theoretic toolkit. To motivate this, consider a ring R. As we have seen, inclusions of points correspond to the natural maps $R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ for prime ideals $\mathfrak{p} \subset R$. Clearly, such a map can be factored as a localization $R \to R_{\mathfrak{p}}$ followed by a quotient $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \to R_{\mathfrak{p}}$, and hence we can ask:

Question. *What is the geometric significance of the localization $R \to R_{\mathfrak{p}}$?*

Note that, although we have already interpreted localizations *at single elements* as inclusions of open subspaces, $R_{\mathfrak{p}}$ is not in general one of these — the complement $R \setminus \mathfrak{p}$ is infinite in many cases of interest, and so we cannot use our usual trick of inverting multiple elements by inverting their product. Hence we should not expect $\operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R$ to be the inclusion of an open subscheme.

To answer our question, first recall from our discussion of fiber products the idea that, for $f, g \in R$, $\operatorname{Spec} R_{fg}$ is the correct "intersection" of $\operatorname{Spec} R_f$ and $\operatorname{Spec} R_g$ in $\operatorname{Spec} R$ precisely because $R_{fg} \cong R_f \otimes_R R_g$ satisfies the universal property of the tensor product — that is, a map $R \to S$ of rings will factor through $R \to R_{fg}$ if and only if it factors through both $R \to R_f$ and $R \to R_g$. (All these factorizations are unique!) Geometrically, this is to say that for a map of schemes $X \to \operatorname{Spec} R$, "the image of X will be contained in $\operatorname{Spec} R_{fg}$" if and only if "it is contained in both $\operatorname{Spec} R_f$ and $\operatorname{Spec} R_g$". (Of course, we have no definition of the *image of a map of schemes* as a scheme, nor does one exist — this is simply a more topologically intuitive way to phrase the idea of a map factoring through inclusions.)

*First draft of the TeX source provided by Márton Beke.

Now observe by the universal property of localizations that a ring map $\phi : R \to S$ will factor through $R \to R_{\mathfrak{p}}$ if and only if, for each $f \notin \mathfrak{p}$, $\phi(f)$ is a unit. This is exactly to say that ϕ factors through each such $R \to R_f$, again by the appropriate universal properties. Since distinguished affine opens form a base for the Zariski topology, we have:

Answer. $\operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R$ *is "the inclusion of the intersection of all open subschemes containing the point corresponding to* $\mathfrak{p}$ *into* $\operatorname{Spec} R$*".*

(For the categorically-minded: All of this is only to say that an intersection is just the inverse limit of a system of inclusions — since $R_{\mathfrak{p}}$, like any localization, can be written as the direct limit of a system of single-element localizations, we should think of its spectrum as the intersection of the corresponding distinguished open subschemes.)

Note that, in the topological spaces we are used to — indeed, in any T_1 space — taking the intersection of all open neighborhoods of a point results in just the point itself, which isn't terribly exciting. In the setting of schemes, however, this recovers quite a lot of information about the point's immediate surroundings — in addition to our usual observation that some data about a scheme isn't captured on the level of points and topological spaces, we have the fact that this intersection contains all points specializing to the chosen one. Indeed, one can show that the prime ideals of $R_{\mathfrak{p}}$ are (naturally identified with) the prime ideals $\mathfrak{q}$ of R such that $\mathfrak{q} \subseteq \mathfrak{p}$.

Often, the data retained by the local ring at a point in $\operatorname{Spec} R$ will even be enough to check whether a given property holds in some neighborhood of the point. In particular, we can often lift propositions from the spectrum of the local ring to a well-chosen distinguished affine open neighborhood; the general form of such arguments is:

1. Prove the statement you want in the local ring $R_{\mathfrak{p}}$.

2. Notice that your proof actually used only finitely many inverses of elements $f_1, \ldots, f_k \notin \mathfrak{p}$ which are not already units in R.

3. Rewrite the proof in the appropriate single-element localization $R_{f_1 \cdots f_k}$.

Of course, this isn't guaranteed to work for *any* statement you might want to prove — nevertheless, it is often a useful approach. You will have a chance to use it for yourself in Exercise 5; for the time being, we limit ourselves to a simple and somewhat silly example:

Example 1. *Suppose we are interested in the statement "there exists a such that* $a^5 \neq a$*", say in the various fields* $\mathbb{Z}/p\mathbb{Z}$ *for prime numbers* $p \in \mathbb{Z}$*. Notice that these fields correspond to closed points of* $\operatorname{Spec} \mathbb{Z}$*, which also has a generic point,* $\operatorname{Spec} \mathbb{Q}$*. Observe also that* $\mathbb{Q} = \mathbb{Z}_{(0)}$ *is the local ring at this generic point.*

We will now use the fact that our statement holds in $\mathbb{Q}$ *to show that it holds in* $\mathbb{Z}/p\mathbb{Z}$ *for all but finitely many primes* p*. Specifically, we can write a proof in* $\mathbb{Q}$ *as follows: "Take* $a = 2 \in \mathbb{Q}$ *and suppose that* $2^5 - 2 = 0$*. Then* $0 = \frac{1}{30} \cdot 0 = \frac{1}{30}(2^5 - 2) = \frac{30}{30} = 1$*, a contradiction."*

Now observe that we really used only two facts about $\mathbb{Q}$*: that* $0 \neq 1$*, which is true in all nonzero rings, and that* 30 *has a multiplicative inverse. Hence we can write virtually the same proof with* $\mathbb{Q}$ *replaced by* $\mathbb{Z}_{30}$ *(N.B.: this denotes the localization, not* $\mathbb{Z}/30\mathbb{Z}$*).*

Moreover, the same proof will work without further modification if we replace $\mathbb{Z}_{30}$ *by any of its nonzero quotient rings, as long as we understand "*$\frac{1}{30}$*" to denote the image of this element of* $\mathbb{Z}_{30}$ *under the quotient map. Hence, since the Zariski topology on the closed points of* $\operatorname{Spec}\mathbb{Z}$ *is the cofinite topology and so the distinguished open affine* $\operatorname{Spec}\mathbb{Z}_{30}$ *contains all but finitely many of them, our statement holds in* $\mathbb{Z}/p\mathbb{Z}$ *for all but finitely many primes* $p \in \mathbb{Z}$*. (Specifically, we can see that it is true for all* $p \notin \{2, 3, 5\}$*.)*

The end result could of course have been proven much more simply in this case, but our focus is the method — for example, it is straightforward to modify this argument to show that, for any nonzero integer polynomial $f : \mathbb{Z} \to \mathbb{Z}$, the corresponding polynomial map $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ vanishes identically for at most finitely many prime numbers p. More broadly, we see that appropriately-formulated statements can pass from a local ring to a distinguished affine neighborhood, as claimed.

So far we have been dealing with local rings at points in affine schemes specifically. To be able to work in general, we verify:

Proposition/Definition 1. *Let* X *be a scheme and* $x \in X$ *a point. Then, if we let* $\operatorname{Spec} R \cong U \ni x$ *be an affine open and* $\mathfrak{p} \subset R$ *the prime ideal corresponding to* x*, the scheme maps* $\operatorname{Spec} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \hookrightarrow \operatorname{Spec} R_\mathfrak{p} \to X$ *are independent of the chosen affine* U*. We call* $\kappa(x) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ *the* **residue field** *of* X *at* x *and* $\mathcal{O}_{X,x} := R_\mathfrak{p}$ *the* **local ring** *of* X *at* x*. (For those with a background in sheaves, the notation is not misleading — the local ring* $\mathcal{O}_{X,x}$ *is indeed the* **stalk** *at* x *of the structure sheaf.)*

In keeping with our perspective that points are just the spectra of fields, we typically write the natural scheme map $\operatorname{Spec}\kappa(x) \hookrightarrow X$ *more simply as the point inclusion* $x \hookrightarrow X$*.*

When combined with our prior observation that properties can often be lifted from the local ring at a point to a neighborhood, this well-definedness allows us to adapt certain ring properties somewhat more sleekly to the context of schemes — rather than requiring that a certain property holds on an affine open cover and then verifying that this is independent of the chosen cover, we can often simply stipulate that it holds on the level of local rings to see automatically that the definition does not depend on choices. Of course, we then typically want to immediately verify that the property is "stalk-local" for rings — that is, that checking it for every local ring of $\operatorname{Spec} R$ is the same as checking it for R, so that our scheme definition actually agrees with the old one for rings — which naturally proves anyway that it is independent of choices when defined on covers.

Remark 1. *We began our study of local rings by wondering, for a ring* R *and prime ideal* $\mathfrak{p}$*, about the geometric meaning of the factorization* $R \to R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$*. Since localizations and quotients commute with one another, we could instead factor the same map as* $R \to R/\mathfrak{p} \to (R/\mathfrak{p})_\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} =: \kappa(\mathfrak{p})$ *and ask about the geometric significance of* $\operatorname{Spec} R/\mathfrak{p}$*. In fact, however, we have already seen the answer to this question in Lecture 2, when we observed that this gave the scheme-theoretic closure in* $\operatorname{Spec} R$ *of our point.*

2 Nilpotents and Reducedness

From our definition of open inclusions for schemes, we can see clearly that an open subscheme of a given scheme is determined by the points it contains — that is, by the corresponding

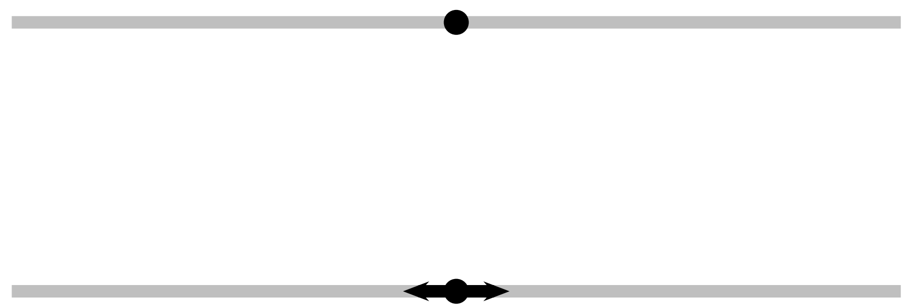

Figure 1: The two subschemes of $\mathbb{A}^1_{\mathbb{C}}$ discussed in Example 2: the point X (above) as compared to the "point with an extra infinitesimal direction" X' (below).

open subset of the underlying topological space. However, as we have already alluded to repeatedly, the same cannot be said for closed subschemes:

Example 2. *Consider the closed subschemes $X = \operatorname{Spec}\mathbb{C}[x]/(x)$ and $X' = \operatorname{Spec}\mathbb{C}[x]/(x^2)$ of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x]$. In each case, the underlying topological space is simply a closed point, the origin, in $\mathbb{A}^1$, but the schemes themselves are not the same.*

To start to obtain a geometric understanding of the difference between the two, consider a polynomial $f = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{C}[x]$. If we restrict this to X — that is, take its image under the quotient map to $\mathbb{C}[x]/(x)$ — the information we retain is precisely the value a_0 of f at the origin, in keeping with our usual philosophy about viewing ring elements as functions whose values at points are given in this way. On the other hand, if we restrict f to X', we are left instead with $a_1 x + a_0$ — that is, the restriction records not only f's value at the origin, but also its first derivative! Hence we think of X' as containing not only the origin but also a sort of "first-order infinitesimal tuft of the x-axis" which is large enough that we can take a derivative along it (albeit only once) but not so large that it actually contains any additional points, closed or otherwise. This situation is depicted in Figure 1.

By instead modding out by higher powers of (x), we could also get "fuzzier points" containing information about higher-order derivatives of polynomials in $\mathbb{C}[x]$ (of only finitely many orders each, however). Of course, there is a problem in our discussion so far — despite the all the talk about "derivatives", we don't yet have any concept of calculus on schemes, and in particular we haven't actually seen a definition of the derivative which makes sense for an element of an arbitrary ring. We will eventually correct this shortcoming — even for the time being, though, it is still reasonable to think informally of the failure of closed subschemes to be determined by their points as arising from the possibility of "extra infinitesimal directions" of the sort seen here.

For a more complicated instance of the same behavior, we consider:

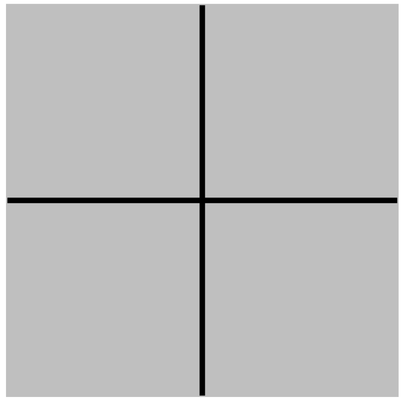

Figure 2: The closed subscheme $\operatorname{Spec}\mathbb{C}[x,y]/(xy)$, as mentioned in Example 3.

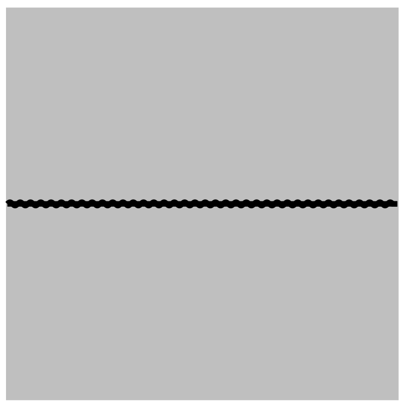

Figure 3: The closed subscheme $\operatorname{Spec}\mathbb{C}[x,y]/(y^2)$, as mentioned in Example 3.

Example 3. *Let X be the closed subscheme* $\operatorname{Spec}\mathbb{C}[x,y]/(xy,y^2)$ *of $\mathbb{A}^2_{\mathbb{C}}$. To understand how to picture X, it's best to think of the quotient in two steps, using either* $\operatorname{Spec}\mathbb{C}[x,y]/(xy)$ *or* $\operatorname{Spec}\mathbb{C}[x,y]/(y^2)$ *as an intermediate. In the former case, we must take the vanishing of the "function" y^2 on the union of coordinate axes depicted in Figure 2; this is to say we cut away all of the y-axis except an infinitesimal tuft.*

In the latter case, alternatively, we have the "fat line" which consists of the x-axis and a bit of first-order infinitesimal tangent information in the y-direction at every point, as depicted in Figure 3, and we take the vanishing of the "function" xy, which has the effect of cutting away this extra tangent fuzz wherever $x \neq 0$.

In either case, the final result will be as depicted in Figure 4 — the underlying set is the x-axis, and there is a single tuft of addition first-order tangent information in the y-direction at the origin.

Note that here we are implicitly building a bit on the idea of ring elements as functions we used to define schemes as ringed spaces — we now think about $\operatorname{Spec} R/I$ (for R a ring, $I \subseteq R$ an ideal) as the "vanishing locus in $\operatorname{Spec} R$ of the functions $f \in I$". This is true literally on the level of underlying spaces — that is, if we take the "value" of a function at a point $\mathfrak{p}$ of

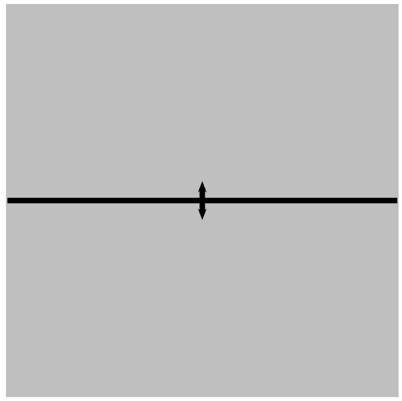

Figure 4: The closed subscheme $\operatorname{Spec} \mathbb{C}[x,y]/(xy,y^2)$ discussed in Example 3.

R to be its image in the residue field $\kappa(\mathfrak{p})$, as discussed in Lecture 2, the underlying space of $\operatorname{Spec} R/I$ is exactly the collection of points where every $f \in I$ "evaluates" to zero. (As we have just seen, of course, the behavior on underlying spaces doesn't give a full description of the scheme structure!) We will develop this idea in greater detail and generality later, when we discuss *ideal sheaves* and their relation to closed inclusions.

Remark 2. *The style of picture-drawing used in Examples 2 and 3 fails to be literally accurate on at least four levels. The first is that, of course, we cannot draw "directional fuzz without points" in a way that genuinely includes no extra points of the picture plane; instead, we indicate the presence of additional infinitesimal directions loosely with arrows, fuzzy lines, and the like, hoping that this will be enough to evoke a useful mental image for the viewer despite the inaccuracy on a strict technical level.*

The second is that we are drawing only the closed points of our subschemes of affine space — strictly speaking, our figures should include extra points corresponding to non-maximal prime ideals. However, since these sorts of non-closed points have no real analogue in the Euclidean world we must perforce embed our pictures in, it is most common to leave them out entirely and trust the viewer to understand roughly "where they are" in the part of the space depicted — that is, what their closures should be.

Relatedly, the third is that even the points we do draw do not really have the topology indicated by the picture — again, we are limited by the necessity of drawing and perceiving images in Euclidean terms, and must simply remember that the open and closed sets are not the ones we would classically expect.

The fourth and final layer of deceit is that the schemes we are considering are defined over $\mathbb{C}$ — following the standard practice, however, we usually draw the vanishings of the corresponding equations over $\mathbb{R}$ instead (sometimes we do so after an appropriate change of coordinates to avoid being too misleading). Drawing a more accurate picture using the topological identification $\mathbb{C} \cong \mathbb{R}^2$ may be possible in some cases — however, the dimensional considerations involved typically make it much less feasible to produce a useful picture in this way.

Having seen some examples of this strange behavior of closed inclusions, we would now like to start to get a handle on when and why it can occur. More particularly, we can ask:

Question. *Let X be a scheme. Under what circumstances will there exist a proper closed subscheme Y of X such that the underlying topological space of Y is the entire underlying topological space of X?*

As is often the case, we can simplify this question by first asking it on the level of *affine* schemes — that is, algebraically.

Question (affine version)**.** *Let R be a ring. When is there a nonzero ideal $I \subseteq R$ such that I is contained in every prime ideal of R?*

Note that here we are implicitly using the identification of prime ideals of R/I with prime ideals of R containing I. It is clear that an ideal with the specified property will exist if and only if there is a nonzero ring element contained in every prime — returning to our idea of ring elements as functions, we can then see that our question amounts to asking about the existence of "a nonzero function which evaluates to zero at every point". Although this runs counter to our classical intuitions about functions, we can see that such elements may exist in general:

Proposition/Definition 2. *Let R be a ring. Then an element $f \in R$ is contained in every prime ideal $\mathfrak{p} \subset R$ if and only if it is* **nilpotent** *(that is, there exists some $n > 0$ such that $f^n = 0$). The ideal $\bigcap_{\mathfrak{p} \subset R \text{ prime}} \mathfrak{p}$ of all such elements is called the* **nilradical** *of R and denoted $\operatorname{nil}(R)$. A ring R such that $\operatorname{nil}(R) = (0)$ is said to be* **reduced***.*

Exercise 1. *Prove that any element contained in every prime ideal is nilpotent, as claimed. (You probably saw proofs of both directions of the claimed equivalence when you learned commutative algebra, but it may be instructive to revisit this one in light of the geometric understanding of the algebraic concepts involved we have developed.)*

We now have an algebraic answer to our question for affine schemes:

Answer (affine version)**.** *Exactly when R is nonreduced.*

Before we generalize to arbitrary schemes, we note that the definition gives a way to make a "reduced version" of any given ring:

Definition 1. *Let R be a ring. Then the* **reduction** *of R is $R_{\text{red}} := R/\operatorname{nil}(R)$.*

$\operatorname{Spec} R_{\text{red}}$ is naturally embedded in $\operatorname{Spec} R$ as a closed subscheme — we can characterize this either as the largest reduced (in the sense of being the spectrum of a reduced ring) closed subscheme or as the smallest closed subscheme with the same underlying topological space as $\operatorname{Spec} R$ itself (because, to get such a closed subscheme, we can mod out only by elements which are zero on every point).

We can now adapt these concepts to the non-affine setting:

Proposition/Definition 3. *Let X be a scheme. Then we define the* **reduction** *X_{red} of X by picking an affine open cover $\{\operatorname{Spec} R_\alpha\}_{\alpha \in A}$ and gluing together the corresponding affine schemes $\{\operatorname{Spec}(R_\alpha)_{\text{red}}\}_{\alpha \in A}$ along the induced identifications; this is well-defined and independent of the chosen cover.*

We say that X is **reduced** *if it satisfies any of the following equivalent properties:*

1. *The only closed subscheme of X containing all points is X itself.*

2. $X = X_{\mathrm{red}}$.

3. *X is affine-locally reduced — that is, there exists an affine open cover of X by the spectra of reduced rings.*

4. *X is stalk-locally reduced — that is, for each point $x \in X$, the local ring $\mathcal{O}_{X,x}$ has no nilpotent elements.*

5. *The structure sheaf $\mathcal{O}_X$ is reduced — that is, for each open subset $U \subseteq X$, the ring $\mathcal{O}_X(U)$ of functions on U has no nilpotent elements.*

The equivalence of these varying definitions of reducedness may lead one to suspect that nilpotence itself should be a local property, but this is not true:

Exercise 2. *Construct a scheme X and global function $f \in \mathcal{O}_X(X)$ such that f is locally nilpotent but not nilpotent — that is, X has an open cover by subspaces U with each restriction $f|_U := \rho_{XU}(f)$ nilpotent, but no power of f itself is zero.*

Bonus: Show that no such example with X affine can be constructed.

We can now answer the original, non-affine version of our question:

Answer. *Exactly when X is nonreduced.*

Hence, although our toolset for understanding this sort of behavior when it occurs is still limited for the time being, we at least have ways of detecting it and, through the reduction, getting rid of it.

Working with reduced schemes thus allows us to close the gap between the notions of "a Zariski-closed subset" and "a closed subscheme":

Proposition 1. *Let X be a scheme and Z a Zariski-closed subset of the underlying space of X. Then there exists a unique reduced closed subscheme of X with underlying set Z.*

In particular, we will often identify a closed subset of X's underlying space with the corresponding reduced subscheme.

3 Irreducibility and Integrality

Having taken our first serious foray into nonreducedness — that is, into the failure of schemes to be determined by their points, as topological spaces are — we now move in the opposite direction, turning our attention to some properties which are more closely linked to the underlying topological space of a scheme.

Definition 2. *We say that a scheme is* **quasicompact** *if every collection of open subschemes which covers it admits a finite subcover.*

Recall that "quasicompactness" for topological spaces is simply the algebro-geometric term for what topologists call "compactness", and observe by the correspondence between open subschemes and open subsets of the underlying topological space that a scheme is quasicompact if and only if its underlying space is. In particular, every affine scheme is quasicompact.

Definition 3. *We say that a scheme is* **connected** *if it cannot be decomposed as the disjoint union of two open subschemes.*

We can see that a scheme is connected if and only if its underlying topological space is, essentially by the reasoning used for quasicompactness.

Exercise 3. *Let R be a ring. Show that $\operatorname{Spec} R$ is connected if and only if R cannot be written as a Cartesian product $R_1 \times R_2$ (with coordinate-wise addition and multiplication) of nonzero rings R_1 and R_2.*

To introduce our next definition, we note that, at least in the topological setting, connectnedness can be reformulated in the following rather unintuitive way:

Proposition 2. *Let X be a topological space. Then X is connected if and only if, whenever we write $X = A \cup B$ for clopen subspaces $A, B \subseteq X$, at least one of A and B is X itself.*

That is, connectedness asserts precisely that a space cannot be nontrivially decomposed as a union (even a non-disjoint one!) of two clopen pieces. Asking that the same be true for *closed* pieces, not just clopen ones, yields the following stronger notion:

Definition 4. *A nonempty topological space X is said to be* **irreducible** *if, whenever we write $X = A \cup B$ for closed subspaces $A, B \subseteq X$, at least one of A and B is X itself. (Equivalently, we can require that every pair of non-empty open subspaces has non-empty intersection, or that every non-empty open subspace of X is dense.)*

We say that a scheme is **irreducible** *if its underlying topological space is.*

Note by the characterization in terms of open subsets that every open subspace of an irreducible space is irreducible.

For the topological spaces we are most used to, this notion is not terribly exciting — a Hausdorff space is irreducible if and only if it contains at most one point. However, as long as we remember that our closed sets are the ones which can be cut out as vanishings of ring elements (that is, if we are working in affine space, by polynomial equations), irreducibility is often pictorially clear:

Example 4. *Compare the curves $Y = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2(x+1))$ and $X = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2)$, as depicted in Figure 5. We can see intuitively that there is no way to break Y into two smaller polynomially-defined pieces, even set-theoretically; on the other hand, it is clear that X's underlying space decomposes as the union of the two constituent lines $\operatorname{Spec} \mathbb{C}[x, y]/(y - x)$ and $\operatorname{Spec} \mathbb{C}[x, y]/(y + x)$. This is to say that Y is irreducible while X is reducible.*

Remark 3. *When defining quasicompactness and connectedness, we stated the natural scheme-theoretic analogues of the corresponding topological properties, then used the correspondence*

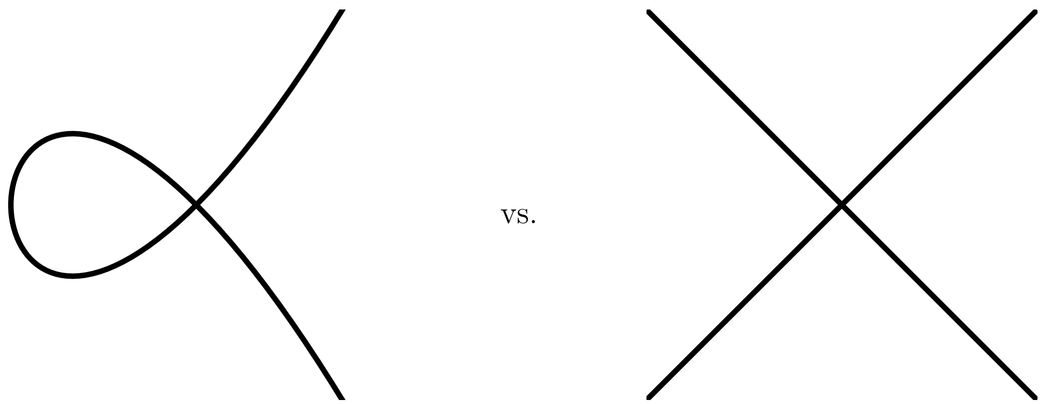

Figure 5: The schemes Y and X, respectively, of Example 4.

between open subsets and open subschemes to observe that requiring either property for a scheme is the same as requiring it for the underlying space. For irreducibility, on the other hand, our definition imposes this equivalence by fiat. It is then natural to wonder if we could instead do what we did before — that is, if we took, say, "irreducibility" of a scheme to mean that every open subscheme is scheme-theoretically dense (which is to say that its inclusion does not factor through the inclusion of any proper closed subscheme), would we get the same result?

Precisely because our original definition involves closed subspaces, whose scheme-theoretic analogues are not topologically determined, this is not the case in general, although it does hold for schemes which are reduced. For example, the scheme $\operatorname{Spec} \mathbb{C}[x,y]/(xy,y^2)$ *of Example 3 is irreducible, but the open subscheme* $\operatorname{Spec}(\mathbb{C}[x,y]/(xy,y^2))_x \cong \operatorname{Spec} \mathbb{C}[x]_x$ *is not dense in the scheme-theoretic sense because it is contained in the proper closed subscheme* $\operatorname{Spec}(\mathbb{C}[x,y]/(xy,y^2))_{\text{red}} = \operatorname{Spec} \mathbb{C}[x,y]/(y) \cong \operatorname{Spec} \mathbb{C}[x]$*. That is, because the failure of reducedness occurs only at the origin, an open subscheme not containing this point will be dense only topologically — or, if one prefers, our scheme can be written nontrivially as the "union" of closed subschemes, say* $\operatorname{Spec} \mathbb{C}[x,y]/(y)$ *and* $\operatorname{Spec} \mathbb{C}[x,y]/(x^2,xy,y^2)$*.*

Hence our definition of irreducibility, although widely accepted in algebraic geometry, is arguably not quite the right one for nonreduced schemes. We will keep to the standard terminology — however, the idea of "the correct scheme-theoretic analogue for irreducibility" will reappear later, when we discuss associated points.

Although we have defined it topologically, irreducibility does admit an algebraic interpretation by way of the reduction:

Proposition 3. *Let* R *be a nonzero ring. Then* $\operatorname{Spec} R$ *is irreducible if and only if* $\operatorname{nil}(R)$ *is prime — that is, if and only if* R_{red} *is an integral domain.*

Proof. Suppose first that $\operatorname{nil}(R)$ is prime, so that R_{red} is a domain. Then, by our discussion in Lecture 2, we can see that $\operatorname{Spec} R_{\text{red}}$ has a generic point — that is, one contained in every open subset of the underlying space. In particular, the closure of each open subset contains the closure of this point, which is the whole space, and so every open set is set-theoretically dense. (Note that, since $\operatorname{Spec} R_{\text{red}}$ is reduced, this is actually the same as being scheme-theoretically dense.) Since $\operatorname{Spec} R$ and $\operatorname{Spec} R_{\mathrm{red}}$ have the same underlying topological space, this proves that $\operatorname{Spec} R$ is irreducible.

Now suppose to the contrary that $\operatorname{nil}(R)$ is not prime. Then we have $f, g \in R$ which are not nilpotent but which have have the property that $fg \in \operatorname{nil}(R)$ — that is, two functions such that neither evaluates to zero on every point, but their product does. Recall for any $h \in R$ that the underlying space of $\operatorname{Spec} R/(h)$ is precisely the collection of points where the "value" of R is zero. In particular, we can see that the underlying space of $\operatorname{Spec} R/(fg)$ will be the union of the underlying sets of $\operatorname{Spec} R/(f)$ and $\operatorname{Spec} R/(g)$, since our function "values" live in the corresponding residue fields and fields are integral domains. The hypothesis that fg vanishes on every point thus implies that the union of the underlying sets of $\operatorname{Spec} R/(f)$ and $\operatorname{Spec} R/(g)$ is the entire underlying space of $\operatorname{Spec} R$. On the other hand, neither of these closed subschemes have underlying set equal to the whole space since neither f nor g evaluates to zero at every point. Hence we have a nontrivial decomposition of our space into two closed pieces, so $\operatorname{Spec} R$ is not irreducible in this case.

The claimed equivalence follows. $\square$

Corollary 1. *A nonempty scheme is irreducible if and only if it has a point which is set-theoretically dense.*

(The proposition shows this for affine schemes — to get the result in general, it essentially remains to show that the irreducibility of the whole space forces the dense points of the subschemes in any affine open cover to be identified under the gluings.)

We have already noted that the notion of irreducibility is best-behaved when the schemes in question are reduced, and this result reinforces that observation — reduced schemes are precisely the ones where set-theoretic and scheme-theoretic density coincide. Hence it is useful to be able to speak concisely about schemes which satisfy both of these properties at once:

Proposition/Definition 4. *A nonempty scheme X is called* **integral** *when it satisfies any of the following equivalent properties:*

1. *X is reduced and irreducible.*
2. *For every open subspace $U \subseteq X$, the ring $\mathcal{O}_X(U)$ of functions on U is an integral domain.*
3. *X has a point which is scheme-theoretically dense — that is, the only closed subscheme containing the point is X itself.*

In particular, we can see for a nonzero ring R that $\operatorname{Spec} R$ is integral if and only if R is an integral domain, as we might hope.

In Lecture 2, we discussed the idea that, for R a ring, each prime $\mathfrak{p} \subset R$ corresponds to the generic point of the irreducible closed subset of $\operatorname{Spec} R$ cut out by $\mathfrak{p}$. We now have a much clearer idea of what this means — the closure of any point in a scheme X will be precisely the integral closed subscheme of which it is the generic point, and by the discussion above every integral closed subscheme of X will have such a generic point. Since every non-irreducible closed subset decomposes into smaller closed pieces, we can think that the integral closed subschemes of X are somehow "the basic building blocks of closed subsets of X", viewed as schemes by endowing them with the natural reduced scheme structure — this allows us to understand our non-closed points by reference to the correspondence of points with such "building blocks".

To make this view more precise, at least in nice cases, it behooves us to introduce...

4 Noetherianity

Recall the following definition from commutative algebra:

Definition 5. *A ring R is said to be* **Noetherian** *if every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of ideals of R is eventually constant. (Equivalently: If every ideal of R is generated by finitely many elements.)*

On the geometric side, this is to say that a ring R is Noetherian precisely when every *descending* chain of closed subschemes of $\operatorname{Spec} R$ is eventually constant.

To generalize this idea to the context of arbitrary schemes, we could simply ask, as we often do, that they possess the property under consideration affine-locally:

Definition 6. *We say that a scheme is* **locally Noetherian** *if it has an affine open cover by spectra of Neotherian rings. (Equivalently: If any affine open subscheme is the spectrum of a Noetherian ring.)*

The parenthetical characterization is enough to verify that this notion is independent of the chosen cover. However, as the terminology suggests, this is not quite the best analogue for Noetherianity in the scheme-theoretic context — instead, we adapt our characterization via closed subschemes:

Proposition/Definition 5. *A scheme is X is said to be* **Noetherian** *if it satisfies any of the following equivalent properties:*

1. *Every descending chain of closed subschemes of X is eventually constant.*
2. *X is locally Noetherian and quasicompact.*
3. *X admits a finite affine open cover by spectra of Noetherian rings.*

That is, we impose an additional finiteness condition to exclude, e.g., infinite disjoint unions of Noetherian ring spectra.

In practice, the overwhelming majority of schemes algebraic geometers work with are Noetherian, or at the very least locally so, and hence we will unabashedly introduce Noetherian hypotheses wherever it simplifies the presentation from this point forward. We observe that such hypotheses are preserved by many basic constructions:

Proposition 4. *Any locally closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.*

Proposition 5. *The local ring of a locally Noetherian scheme at any point is Noetherian.*

Proposition 6. *Any (locally) finite-type scheme over a (locally) Noetherian scheme is (locally) Noetherian.*

(Viewed algebraically, this last proposition is just the assertion that any finitely-generated algebra over a Noetherian ring is Noetherian.)

As with many of the concepts we have been discussing, there is a topological version of Noetherianity. As one might expect from the scheme-theoretic definition:

Definition 7. *A topological space is called* **Noetherian** *if any descending chain of closed subspaces of X is eventually constant.*

Like irreducibility, this notion is not especially interesting for the sorts of topological spaces we are used to encountering — a Noetherian Hausdorff space must be finite.

As one would hope, the relationship between topological and scheme-theoretic Noetherianity is well-behaved:

Proposition 7. *The underlying topological space of a Noetherian scheme is Noetherian.*

However, the converse is not true — Noetherianity of the underlying space does not guarantee the Noetherianity of a scheme. The most immediate approach is to use the nonreduced structure not captured on the level of underlying spaces:

Example 5. *Let k be a field. Then the underlying topological space of the infinitely-generated k-algebra $\operatorname{Spec} k[x_1, x_2, \ldots]/({x_1}^2, {x_2}^2, \ldots)$ is a single point, but this scheme is not Noetherian.*

Of course, we can ask whether nonreducedness is the only obstruction to this implication, and indeed it seems like it might be so, since the difference between the concepts "a descending chain of closed subschemes" and "a descending chain of closed subsets" is precisely the possibility of nonreduced behavior. However, as we can see by Examples 2 and 3, a closed subscheme of a reduced scheme is by no means necessarily reduced, so it is not clear how to turn this idea into a proof. This lack of clarity stems from the fact that the statement we are attempting to prove is false:

Example 6 (due to Karl Schwede). *The subring $\mathbb{C}[x, xy, xy^2, xy^3, \ldots]$ of $\mathbb{C}[x, y]$ is not Noetherian, but its spectrum has Noetherian underlying space.*

Exercise 4 (very optional). *Verify this. (Hint: It may be helpful to view our ring $R = \mathbb{C}[x, xy, xy^2, xy^3, \ldots]$ as a quotient of an appropriately-chosen polynomial ring in infinitely many variables, and to show that the map $\mathbb{A}^2_{\mathbb{C}} \to \operatorname{Spec} R$ induced by the ring inclusion is surjective on underlying spaces.)*

In any event, we bring up the Noetherianity of topological spaces largely to gain access to the following definition:

Proposition/Definition 6. *Let X be a Noetherian topological space. Then there is a unique (up to reordering) decomposition $X = X_1 \cup \ldots \cup X_k$ of X as the union of finitely many irreducible closed subspaces such that $i \neq j$ implies $X_i \not\subseteq X_j$ for all $1 \leq i, j \leq k$. These X_i are called the* **irreducible components** *of X.*

If X is a Noetherian scheme, we will refer to the irreducible components of its underlying space as the **irreducible components** *of the scheme X itself; generally we will regard these as closed subschemes by endowing them with the natural reduced scheme structure.*

Hence, in the Noetherian setting, our characterization of the integral closed subschemes of a scheme as the "basic building blocks of closed subsets of its underlying space" is true in the fairly literal sense that any such closed subset is the (potentially non-disjoint) union of finitely many such pieces — uniquely so if we require that this union contain no redundancies.

In the case of an affine scheme, the irreducible components can be characterized algebraically:

Proposition 8. *Let R be a Noetherian ring. Then the irreducible components of* $\operatorname{Spec} R$ *are given by the closed subschemes* $\operatorname{Spec} R/\mathfrak{p}$ *for minimal prime ideals $\mathfrak{p} \subset R$ (i.e., those containing no smaller prime ideal).*

For example, the reducible scheme $X = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2)$ discussed in Example 4 has as its irreducible components the lines $\operatorname{Spec} \mathbb{C}[x, y]/(y - x)$ and $\operatorname{Spec} \mathbb{C}[x, y]/(y + x)$, and one can verify that $(y - x)$ and $(y + x)$ are indeed the minimal prime ideals of $\mathbb{C}[x, y]/(y^2 - x^2)$.

Now that we have the machinery of Noetherianity, we conclude with our promised exercise on lifting statements from local rings to open neighborhoods:

Exercise 5. *Let X be a locally Noetherian scheme and $x \in X$ a point such that the local ring $\mathcal{O}_{X,x}$ is reduced. Show that x has a reduced affine open neighborhood in X.*