The Geometry of Rings and Schemes
Lecture 5: Dimension Theory and Singular Points

Alex Hof*

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Last week, we discussed some features of the geometry of schemes which lack ready analogues in the most familiar topological settings. Of particular interest for our ongoing discussion will be the ideas that, at least for Noetherian schemes, (the underlying sets of) integral closed subschemes form the basic building blocks out of which all closed subsets are formed (by taking unions), and that, for each such integral closed subscheme, we have a "generic point" whose closure is precisely that subscheme.

We now turn to a notion which *does* have an analogue in the world of, say, differential geometry, but which is difficult to formalize by direct analogy: dimension. This difficulty arises from the fact that, for manifolds, we have local isomorphisms everywhere to objects (open subsets of Euclidean space) for which the right notion of dimension is obvious — on the other hand, schemes are locally isomorphic only to ring spectra, a much more varied class of object for which the "correct" definition of dimension is already unclear. In the setting of classical algebraic geometry — for example, for polynomially-defined subsets of real or complex Euclidean space — the outlook is not quite as bleak; we can think of removing the *singular points*, those at which our set is not already a manifold, and computing the dimension of the remainder. However, this too generalizes poorly (e.g., to $\operatorname{Spec}\mathbb{Z}$), and in fact we will see that for schemes it is more expeditious to define singular points in terms of dimension than the reverse. Instead, we take advantage of the exotic features discussed last week to formalize dimension in a way which, while unfamiliar at first blush, ultimately captures our basic intuitions.

(For a historical account of the notion of dimension in algebraic geometry, see Chapter 8 of Eisenbud's *Commutative Algebra with a View Toward Algebraic Geometry*.)

1 Krull Dimension

Let us begin by calibrating our intuitions a bit. As we know, the points in the world of schemes are the spectra of fields — as such, we should expect "$\dim \operatorname{Spec} k = 0$" for k a field. More generally, we have seen for $n \geq 0$ and k a field that $\mathbb{A}^n_k$ should be thought of as "the geometric realization of the vector space k^n", which is to say Euclidean n-space over k, and hence we should expect "$\dim \mathbb{A}^n_k = n$".

*First draft of the TeX source provided by Márton Beke.

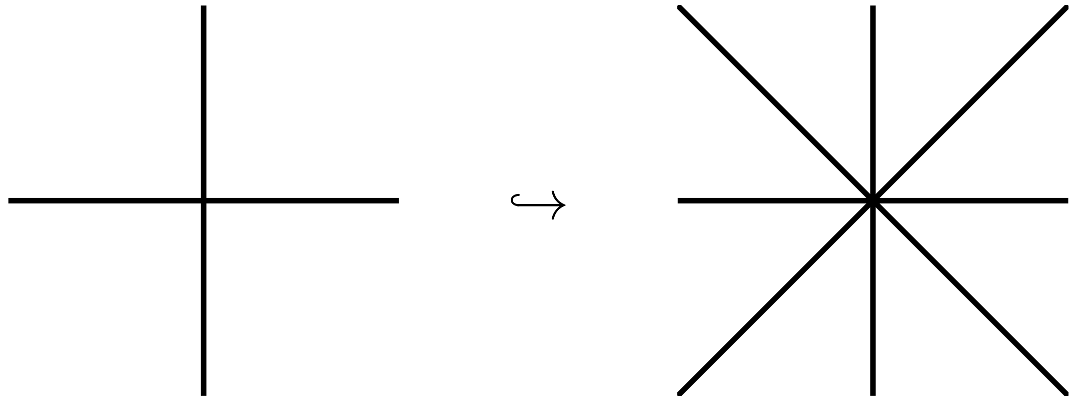

Figure 1: Two schemes which should probably be one-dimensional, all things considered.

Now let X be a scheme and $Y \hookrightarrow X$ a proper closed subscheme. It is fairly clear that, for any reasonable notion of dimension, we should have "$\dim Y \leq \dim X$" — that is, we should not be able to decrease a scheme's dimension by enlarging it. This naturally invites:

Question. *When should and shouldn't we expect that "$\dim Y < \dim X$"?*

Even by considering Euclidean space over $\mathbb{C}$ or $\mathbb{R}$, we have reason to expect that the answer should be "fairly often" — if we take the vanishing locus of even a single polynomial in $\mathbb{C}^n$ or $\mathbb{R}^n$, we can see that the result should be a closed subset of strictly smaller dimension in the sense mentioned above. (By contrast, the classical topology is rife with closed subsets of the same dimension as the ambient space — consider a closed ball, e.g.) Nevertheless, there are some algebro-geometric circumstances where a closed subscheme should be considered full-dimensional:

Answer (partial)**.** *If Y has all the same points as X — for example, $Y = X_{\mathrm{red}}$ — we should think that "$\dim Y = \dim X$".*

That is, in keeping with our idea that nonreduced structure reflects something insubstantial, something infinitesimally small, we do not take its addition or removal to change the dimension of a scheme. Hence we can limit our focus to the case where X is reduced, and ask our question again in this context.

Here we still have situations where we should want equality:

Answer (very partial)**.** *If X is not integral, we should think that there is at least some possibility that "$\dim Y = \dim X$" — after all, we could, e.g., remove some irreducible components from a Noetherian scheme in such a way that the dimension of the remaining ones should still be as large. (See Figure 1 for an example.)*

Let us consider for a moment, then, the case where X is integral. In this situation, as we have seen, Y cannot have all the same points as X, so its underlying set has some nonempty open complement — and this, being a nonempty open subset of the underlying space of an integral scheme, is dense. (Because it contains the generic point, if you'd like.) This is to say that Y is in some sense quite small relative to X, since its complement is large enough to have closure the whole space — if you prefer, you can also note that Y has empty interior. Hence:

Answer (still partial)**.** *If X is integral, we should expect that "$\dim Y < \dim X$".*

Even if we drop the integrality hypothesis on X itself, this viewpoint still suggests something about the dimension of X. Specifically, suppose that we have a chain $Y_0 \subset Y_1 \subset \ldots \subset Y_k$ of (distinct) integral closed subschemes of X. Then, precisely because the Y_i are integral, we should think that "$\dim X \geq \dim Y_k \geq \dim Y_{k-1} + 1 \geq \ldots \geq \dim Y_0 + k \geq k$" (since we should want our schemes, at least nonempty ones like Y_0, to have nonnegative dimension).

This provides the groundwork for our formalization of the notion of dimension:

Definition 1. *Let X be a nonempty scheme. Then the* **dimension** *of X is*

$$\dim X := \max\{k \geq 0 \mid \exists Y_0 \subset Y_1 \subset \ldots \subset Y_k \textit{ integral closed subschemes of } X\},$$

where we take the maximum to be ∞ if the set in question is not bounded above.

If R is a ring, then $\dim R := \dim \operatorname{Spec} R$ is called the **Krull dimension** *of R.*

Our prior discussion tells us that this quantity should at least be a lower bound on the "dimension" of a scheme — the intuition for the idea that it should be an upper bound as well is subtler. At least in the Noetherian setting, where we have the decomposition of any closed subset into irreducible components, this is a strengthening of the idea that we should expect a space to have closed subspaces of all smaller dimensions — if we think of smooth manifolds, for instance, we can observe that it is not and should not be possible to construct a two-dimensional manifold without any one-dimensional closed submanifolds. More pragmatically, one can observe by way of justification that this is the notion which turns out to match our intuitions for affine space over a field, for example.

Example 1. *Let $X = \mathbb{A}^2_{\mathbb{C}}$. Set $Y_0 = \operatorname{Spec} \mathbb{C}[x,y]/(x,y)$, $Y_1 = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3)$, and $Y_2 = X$. Then, since the corresponding rings are domains, we can see that these closed subschemes are all integral, and we have $Y_0 \subset Y_1 \subset Y_2$. This situation is depicted in Figure 2.*

Hence $\dim \mathbb{A}^2_{\mathbb{C}} \geq 2$, as expected; to show this is an equality, we can either remember our description of the points of $\mathbb{A}^2_{\mathbb{C}}$ from Lectures 1 and 2, or use the machinery which we will develop later on in this lecture.

Through the correspondence between integral closed subschemes and their generic points, we can also phrase this in terms of specializations of points:

Proposition 1. *Let X be a nonempty scheme. Then*

$$\dim X = \max\{k \geq 0 \mid \exists p_0, \ldots, p_k \in X \textit{ distinct points such that } \overline{p_{i-1}} \ni p_i \ \forall 1 \leq i \leq k\}$$

(where we again take the maximum to be ∞ if the set is unbounded).

The proof is by taking generic points of chains of integral closed subschemes, in one direction, and taking closures of chains of points, in the other — be mindful of the reversal in indexing between the chains used in the two definitions.

From this proposition or the definition itself, we find:

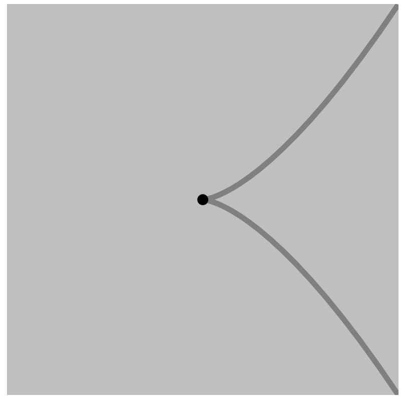

Figure 2: The closed subschemes $Y_0 \subset Y_1 \subset Y_2$ of Example 1.

Proposition 2. *Let R be a ring. Then*

$$\dim R = \max\{k \geq 0 \mid \exists \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_k \textit{ prime ideals of } R\}$$

(with the usual caveat about ∞).

This was the original definition of Krull dimension. Note that the indexing agrees with that of the prior proposition, rather than the definition.

Remark 1. *Either from the original definition or from the characterization in terms of specializations of points, we can see that the dimension of a scheme depends only on its underlying space. In general, for topological spaces, we can define a corresponding notion of dimension by replacing "integral closed subschemes" in our definition with "irreducible closed subsets" — the dimensions of a scheme and its underlying space will then agree. (Note that, as with the idea of irreducible sets itself, this is not a very useful concept in the setting of Hausdorff spaces.)*

In the Noetherian setting, where we can realize a scheme's underlying space as the finite union of irreducible pieces, we can see that the dimension, as we would expect, is just the dimension of the largest piece:

Proposition 3. *Let X be a nonempty Noetherian scheme. Then*

$$\dim X = \max\{\dim Y \mid Y \textit{ is an irreducible component of } X\}.$$

(In the case of ring spectra, it may be useful here to remember our correspondence between irreducible components and minimal primes — the key is that any integral closed subscheme will be contained in one or more of the irreducible components.)

Example 2. *Let $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xz, yz)$ be the reduced subscheme of $\mathbb{A}^3_{\mathbb{C}}$ whose underlying space is the union of the xy-plane with the z-axis, as depicted in Figure 3.*

The prime ideals of $\mathbb{C}[x, y, z]/(xz, yz)$ are (canonically identified with) the prime ideals of $\mathbb{C}[x, y, z]$ containing the ideal (xz, yz); it is clear that these are precisely the prime ideals

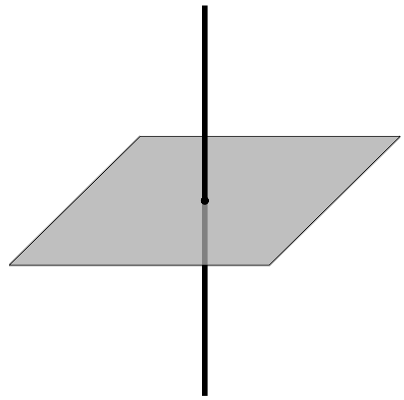

Figure 3: The scheme X of Examples 2, 3, 4, and 5.

which contain either (z) or (x,y). Since the ideals (z) and (x,y) are prime, it follows that they are the minimal primes of $\mathbb{C}[x,y,z]/(xz,yz)$ — hence our irreducible components are the xy-plane (the vanishing of (z)) and the z-axis (the vanishing of (x,y)), as expected.

Now, the primes containing (z) correspond to points of $\operatorname{Spec}\mathbb{C}[x,y,z]/(z) \cong \operatorname{Spec}\mathbb{C}[x,y] \cong \mathbb{A}^2_{\mathbb{C}}$ and the primes containing (x,y) correspond to points of $\operatorname{Spec}\mathbb{C}[x,y,z]/(x,y) \cong \operatorname{Spec}\mathbb{C}[z] \cong \mathbb{A}^1_{\mathbb{C}}$ — these correspondences preserve the specialization relations. Hence the poset of primes of $\mathbb{C}[x,y,z]/(xz,yz)$ under containment is obtained from those of $\mathbb{C}[x,y]$ and $\mathbb{C}[z]$ precisely by taking the disjoint union and then identifying the maximal ideals ((x,y) and (z), respectively) which correspond to the point of intersection of the two irreducible components (i.e., the closed point cut out by (x,y,z) in the original ring).

From this, we can see that a longest chain of primes will be a longest such chain contained in either irreducible component, and that this maximal length will be 2, coming from the irreducible component cut out by (z). Hence $\dim X = 2 = \max\{2,1\} = \max\{\dim \mathbb{A}^2_{\mathbb{C}}, \dim \mathbb{A}^1_{\mathbb{C}}\}$, as we should expect.

It is fairly clear that the scheme of the preceding example should be globally 2-dimensional, since it is the union of a 2-dimensional object with a 1-dimensional one. However, if we pick a small enough open neighborhood of a point on the z-axis away from the origin, we find that this neighborhood should be 1-dimensional; we would like to be able to formalize this idea of "the dimension around a point". Fortunately, we already have a construction, our "intersection of all open subschemes containing a point", which intrinsically captures facts about sufficiently small open neighborhoods of a point:

Definition 2. *Let X be a scheme and $x \in X$ a point. Then we call the Krull dimension $\dim_x X := \dim \mathcal{O}_{X,x}$ of the local ring at x the* **(local) dimension** *of X at x.*

We can verify that this gives the expected result in the case under consideration:

Example 3. *We return to the scheme $X = \operatorname{Spec}\mathbb{C}[x,y,z]/(xz,yz)$ discussed in Example 2 and depicted in Figure 3; let $R = \mathbb{C}[x,y,z]/(xz,yz)$. We will compute the local dimensions of X at its closed points to verify that the result is as expected.*

First consider a closed point on the z-axis away from the origin — this is cut out by a maximal ideal of the form $\mathfrak{m} = (x, y, z - \gamma)$ for $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Since z is in the complement of such an ideal and introducing an inverse for z necessarily kills x and y in R (e.g., $x = xzz^{-1} = (xz)z^{-1} = 0z^{-1} = 0$), we find that these elements are in the kernel of the localization map $R \to R_{\mathfrak{m}}$. Hence we can see that $R_{\mathfrak{m}} \cong \mathbb{C}[z]_{(z-\gamma)}$; this has Krull dimension 1, so the dimension of X at such a point is 1, as desired.

Now consider a closed point on the xy-plane away from the origin, which will be cut out by a maximal ideal of the form $\mathfrak{m} = (x - \alpha, y - \beta, z)$ for $\alpha, \beta \in \mathbb{C}$ with $(\alpha, \beta) \neq (0, 0)$. Then, in particular, at least one of x and y is not contained in $\mathfrak{m}$, so z is in the kernel of the localization map $R \to R_{\mathfrak{m}}$. As such, we can see that $R_{\mathfrak{m}} \cong \mathbb{C}[x, y]_{(x-\alpha, y-\beta)}$, which has Krull dimension 2 (to show this, recall that the prime ideals of this ring are in bijection with the prime ideals of $\mathbb{C}[x, y]$ contained in $(x - \alpha, y - \beta)$). Hence the local dimension at such a point is 2, again as expected.

Finally, we can see that the local dimension at the origin (cut out by (x, y, z)) is 2, essentially by following a modified version of the reasoning of Example 2 or more simply by applying Proposition 3.

We will defer our discussion of the local dimensions at non-closed points until later, when we have a better grasp of their meaning.

Armed with this machinery, we can now verify that dimension is in some sense a local property — specifically, the dimension of a scheme is its largest dimension at any point:

Proposition 4. *Let X be a nonempty scheme. Then $\dim X = \max\{\dim_x X \mid x \in X\}$. If X is moreover Noetherian or finite-dimensional, then in fact $\dim X = \max\{\dim_x X \mid x \in X \text{ is a closed point}\}$.*

This is perhaps easiest to see using the definition of dimension in terms of finite chains of specializing points — if x is the point in such a chain contained in the closure of all the others, we can see that they will be contained in every open neighborhood of x; in particular, there is a corresponding chain of prime ideals in $\mathcal{O}_{X,x}$. To check dimensions only at closed points, the essential condition we need is that X contain no infinite descending chain of *integral* closed subschemes — this is evident a fortiori if X is Noetherian, and can be seen directly if X is supposed finite-dimensional.

From the proposition, it follows that dimension is not only "stalk-local" in this sense, but indeed affine-local; that is, we could have defined dimension by the old standby of first defining it for affine schemes (i.e., for rings), then stitching together over affine open covers in the appropriate sense (in this case, by taking the maximum of dimensions on affine patches). (However, since the Krull dimension is not particularly intuitive in any way independent of the geometry, the advantages of this approach are limited.)

The proposition also allows us to more easily verify a basic dimensional intuition:

Exercise 1. *Let X be a nonempty scheme. Show, for Y a nonempty locally closed subscheme of X, that $\dim Y \leq \dim X$.*

We conclude our initial discussion of dimension by remarking on some seemingly pathological behavior. The phrase "Noetherian or finite-dimensional" in Proposition 4 may initially strike one as odd — after all, shouldn't Noetherian and finite-dimensional be the same thing? (Or, at the very least, shouldn't one imply the other?)

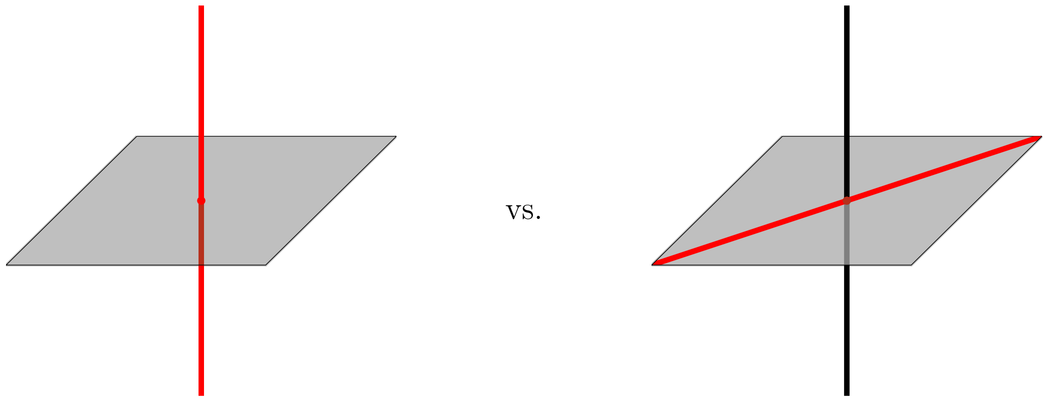

Figure 4: The subschemes Y and Z, respectively, of X in Example 4.

Exercise 2. *Show that the local ring* $\mathbb{C}[x, x^{1/2}, x^{1/3}, x^{1/4}, \ldots]_{(x, x^{1/2}, x^{1/3}, x^{1/4}, \ldots)}$ *is non-Noetherian and has Krull dimension 1.*

Hence there are finite-dimensional rings which are not Noetherian; it turns out to also be true that there are Noetherian rings which are infinite-dimensional. (For one example, due to Nagata, see Exercise 12.1.M of Vakil.) However, such pathologies are rarely encountered in practice.

2 Codimension

We come now to the notion of *codimension*. For those with a background in differential geometry, it may seem odd that we should need to treat this separately from dimension — for a manifold, after all, the codimension of a closed submanifold is simply the difference in dimensions. However, in the more permissive (in some senses) context of schemes, this quantity is not quite as useful:

Example 4. *We return again to the scheme* $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xz, yz)$ *of Examples 2 and 3. Consider the integral closed subschemes given by* $Y = \operatorname{Spec} \mathbb{C}[x, y, z]/(x, y)$ *and* $Z = \operatorname{Spec} \mathbb{C}[x, y, z]/(x - y, z)$*. These are both 1-dimensional integral subschemes of a 2-dimensional scheme but, as we can see in Figure 4, there is a qualitative difference in how they are embedded — despite technically having lower dimension than* X*,* Y *is "mostly full-dimensional", while* Z *has strictly smaller dimension than its surroundings everywhere.*

To rectify the issue, we return to the idea that dimensions — and now, "jumps in dimension" in the sense we will want for codimension — should be witnessed completely by chains of (integral) closed subschemes; for example, if we have an integral subscheme we want to think of as having "codimension 2", we should expect it to be contained in an integral subscheme of "codimension 1", with no possibility that we can jump between "codimension 2" and "codimension 0" without such an intermediate step. That is, we define:

Definition 3. *Let X be a scheme and Y an integral closed subscheme. Then the* **codimension** *of Y in X is*

$$\operatorname{codim}_X Y := \max\{k \geq 0 \mid \exists Y = Y_0 \subset Y_1 \subset \ldots \subset Y_k \text{ integral closed subschemes of } X\}$$

(or, as usual, ∞ if the set is unbounded).

If, instead, X is a scheme and Y is an arbitrary nonempty closed subscheme, we define the **codimension** *of Y in X by setting*

$$\operatorname{codim}_X Y := \min\{\operatorname{codim}_X Z \mid Z \text{ an integral closed subscheme of } Y\}.$$

That is, we modify the definition of dimension by considering only chains of integral closed subschemes ascending from Y. Observe that the definition of the codimension of an arbitrary closed subscheme is well-formed; a priori one might worry about being able to construct a scheme so pathological that it has no integral closed subschemes, but the correspondence between integral closed subschemes and points shows that this cannot occur if the scheme is nonempty. It is also worth noting that, in practice, this definition in the non-integral case is perhaps not so useful — to avoid confusion, it will generally be best just to think directly about the different irreducible components of a subscheme (in the Noetherian case, where these are well-defined) and consider their codimensions individually.

As in the case of dimension, and by essentially the same reasoning, we have ideal-theoretic and point-theoretic descriptions of codimension:

Proposition/Definition 1. *Let R be a ring and $\mathfrak{p} \subset R$ a prime ideal. Then*

$$\operatorname{codim}_{\operatorname{Spec} R}(\operatorname{Spec} R/\mathfrak{p}) = \max\{k \geq 0 \mid \exists \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_k = \mathfrak{p} \text{ prime ideals of } R\}$$

is the quantity which, in commutative algebra, is called the **height** *of $\mathfrak{p}$.*

More generally, let X be a scheme and Y an integral closed subscheme with generic point η. Then

$$\operatorname{codim}_X Y = \{k \geq 0 \mid \exists p_0, \ldots, p_k = \eta \in X \text{ distinct points such that } \overline{p_{i-1}} \ni p_i \ \forall 1 \leq i \leq k\}.$$

As a consequence, we find that $\operatorname{codim}_X Y = \dim_\eta X := \dim \mathcal{O}_{X,\eta}$ is the local dimension of X at the generic point of Y.

This latter characterization sheds some light on both the idea of codimension and the meaning of local dimension at a non-closed point. In particular, we find that what we call "codimension" might be better termed "*generic* codimension" — it is now clear that this concept ignores anything that happens on any particular proper closed subset of the integral closed subscheme being considered. For instance, we see that it should not matter that the subscheme Y of Example 4 is embedded into X with a positive difference in dimension *at the origin*; since the origin is a proper closed subset of Y, we can ignore this behavior and note that Y is full-dimensional in X everywhere else along its length. Hence its codimension should be zero, which turns out to be the case:

Example 5. *Let* $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xz, yz)$ *as in Examples 2, 3, and 4 (see Figure 3) and recall in particular our description of the points of* X *in Example 2. We will now compute the codimensions of the corresponding integral closed subschemes. Let* $R = \mathbb{C}[x, y, z]/(xz, yz)$*.*

The codimensions of the closed points are, by Proposition/Definition 1, simply the local dimensions of X *at the corresponding points, which we computed in Example 2. (It is a good idea to go back and check that this agrees with your intuitive notion of the "codimension of a point".)*

The generic point of the z*-axis corresponds to the prime ideal* (x, y)*. Since this is one of the minimal primes of* R*, it has height 0; hence the codimension of the* z*-axis is 0 as expected by Proposition/Definition 1.*

Likewise, since the generic point (z) *of the* xy*-plane is a minimal prime of* R*, we find that the* xy*-plane has codimension 0.*

Finally, we compute the codimension of an integral curve in the xy*-plane — these will have generic points corresponding to the non-maximal primes* $\mathfrak{p}$ *containing* (z)*. We can see such primes have height 1 (witnessed by the chain* $(z) \subset \mathfrak{p}$*), so these curves have codimension 1. Thus, in particular, codimension gives a quantification of the qualitative difference described in Example 4 and depicted in Figure 4.*

Here we have seen some behavior which we might as well write down in general:

Proposition 5. *Let* X *be a Noetherian scheme and* Y *an integral closed subscheme. Then* Y *is an irreducible component of* X *if and only if* $\operatorname{codim}_X Y = 0$*.*

That is, Y is an irreducible component if and only if it is "set-theoretically all of X generically on Y". More broadly, we can think that codimension captures some kind of "generic difference in dimension" in general — however, some caution is needed, since this has unexpected consequences, and one can produce, e.g., an irreducible Noetherian scheme of dimension 2 with a codimension-1 integral closed subscheme of dimension 0 (see Vakil Section 12.3.13). In any case, we will proceed with some rough intuition and a healthy dose of caution about applying it too freely.

3 Krull's Height Theorem

In the Noetherian setting, the following useful theorem gives some useful algebraic control on the possible codimension(s) of (the irreducible components of) a subscheme defined as the "vanishing locus" of a collection of "functions" — for simplicity, we treat the affine case:

Theorem 1 (Krull's Height Theorem)**.** *Let* R *be a Noetherian ring and* $f_1, \ldots, f_c \in R$*. Let* Z *be an irreducible component of* $\operatorname{Spec} R/(f_1, \ldots, f_c)$*. (I.e.,* $Z = \operatorname{Spec} R/\mathfrak{p}$ *for* $\mathfrak{p} \subset R$ *minimal among primes containing* $(f_1, \ldots, f_c)$*.)*

Then $\operatorname{codim}_{\operatorname{Spec} R} Z \leq c$*.*

The $c = 1$ case is called "Krull's Principal Ideal Theorem", or, to sound fancy, "Krull's *Hauptidealsatz*" (this is German for "Principal Ideal Theorem"). Some authors may also refer to the general result by this terminology, even though it no longer specifically involves a principal ideal. The basic idea is that "cutting out by 1 equation" should "reduce the

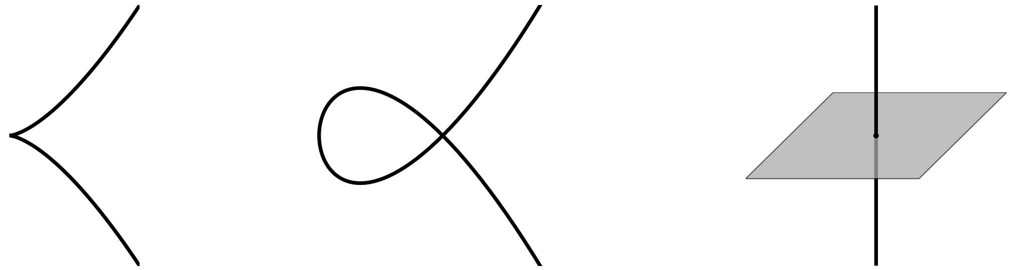

Figure 5: Some schemes in affine spaces which are singular at the origin and nonsingular elsewhere — respectively, $\operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^3)$, $\operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^2(x+1)$, and $\operatorname{Spec}\mathbb{C}[x,y,z]/(xz,yz)$.

dimension" by at most 1, or more generally that "cutting out by c equations" should "reduce the dimension" by at most c — of course, we know by this point that the meaning of the bounds on codimension here is somewhat more subtle than the literal interpretation of these statements would suggest.

The proof of Krull's Height Theorem uses some machinery we have yet to cover, so we will not go into the details for now. Instead, we apply it to refine our understanding of dimension:

Exercise 3. *Use Krull's Height Theorem to show that, if R is a Noetherian local ring with maximal ideal $\mathfrak{m}=(f_1,\ldots,f_d)$, $\dim R\le d$.*

Hence, although we have already noted that Noetherian rings may be infinite-dimensional in general, we can see that Noetherian *local* rings exhibit no such pathology.

We can also confirm at long last that our definition of dimension gives the expected result for affine space:

Exercise 4. *Let k be a field and $n\ge 0$ an integer. Use the result of the preceding exercise (and some other stuff from this lecture) to prove that $\dim\mathbb{A}^n_k=n$.*

4 Regularity

As mentioned, our concept of dimension will allow us to take a first look at the idea of singular and nonsingular points — that is, points where a given scheme doesn't or does "look like a manifold". (For some examples of this behavior, see Figure 5.)

We will capture this phenomenon by returning to the notion of "infinitesimal directions at a point" which came up when we discussed nonreducedness, and in particular the idea that these "directions" correspond somehow to nilpotent elements:

Definition 4. *Let X be a scheme, $x\in X$ a point, and $k\ge 0$ an integer. Then, for $R=\mathcal{O}_{X,x}$ the local ring at x and $\mathfrak{m}$ its maximal ideal, we call $\operatorname{Spec} R/\mathfrak{m}^{k+1}$ the **kth-order infinitesimal neighborhood** of x in X.*

The idea is that elements of our local ring R should be thought of as "functions defined locally near x on X" (that is, for those who know the term, *germs* of functions) and $\mathfrak{m}^{k+1}$ should be thought of as "the collection of such local functions vanishing to order at least $k+1$ at x". That is, what we get by restricting a function to the infinitesimal neighborhood $\operatorname{Spec} R/\mathfrak{m}^{k+1}$ should be thought of as "its value and derivatives of orders $\leq k$ at x". (This is an extension of our idea that restricting to the 0th-order infinitesimal neighborhood, $x = \operatorname{Spec} \kappa(x) = \operatorname{Spec} R/\mathfrak{m}$ itself, gives the value of a function at x.) Note that all infinitesimal neighborhoods of a point have underlying space simply the point itself — the additional behavior is entirely on the level of nonreduced structure.

It may be easiest to see this behavior through a concrete example:

Example 6. *Let $X = \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$ and take x to be the origin. Then our local ring is $R = \mathbb{C}[x,y]_{(x,y)}$, with maximal ideal $\mathfrak{m} = (x,y)$. Let $f = \sum_{i,j} a_{ij}x^iy^j \in \mathbb{C}[x,y]$ be a polynomial.*

As we know, the image of f under restriction to the 0th-order infinitesimal neighborhood $\operatorname{Spec} R/(x,y) \cong \operatorname{Spec} \mathbb{C}$ is simply a_{00}, its value at the origin.

Now consider the restriction to the 1st-order infinitesimal neighborhood $\operatorname{Spec} R/(x^2,xy,y^2)$. f is sent to the element $a_{00} + a_{10}x + a_{01}y$, from which we can recover both the value and all first-order directional derivatives of f at the origin.

Likewise, if we instead restrict our function f to the 2nd-order infinitesimal neighborhood $\operatorname{Spec} R/(x^3,x^2y,xy^2,y^3)$, the result is $a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2$, which tells us both all of the previous information and the data of the 2nd-order derivatives (if you want, the Hessian matrix) of f at the origin.

Restriction to higher-order infinitesimal neighborhoods behaves similarly.

At least for now, we will be interested mainly in the case $k = 1$ — that is, the ring $R/\mathfrak{m}^2$. As noted, its spectra is the 1st-order infinitesimal neighborhood of the point, capturing values and first-order derivatives of functions. Considering its maximal ideal $\mathfrak{m}/\mathfrak{m}^2$, which consists of "functions" on the infinitesimal neighborhood with value zero at x, then allows to get at just the derivative information without worrying about values:[1]

Definition 5. *Let X be a scheme, $x \in X$ a point, and $\mathfrak{m} \subset \mathcal{O}_{X,x}$ the maximal ideal of the local ring. Then $\mathfrak{m}/\mathfrak{m}^2$ is the* **Zariski cotangent module** *of X at x; its elements are called* **cotangent vectors** *at x.*

The "vector" terminology is justified by the fact that the $\mathcal{O}_{X,x}$-action by multiplication on $\mathfrak{m}/\mathfrak{m}^2$ factors through the quotient $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m} =: \kappa(x)$ (since multiplying $\mathfrak{m}$ by any element of $\mathfrak{m}$ gives a result which is zero modulo $\mathfrak{m}^2$) — hence the Zariski cotangent module is really a vector space over the field $\kappa(x)$. Indeed, the standard terminology reflects this:

Remark 2. *The "Zariski cotangent module" terminology is slightly nonstandard. Most authors instead call this object the "Zariski cotangent space" of X at x, and even go so far as to define a "Zariski tangent space" by taking its dual as a $\kappa(x)$-vector space. However, the use of the term "space" to describe a purely algebraic object should make you uneasy, and reflects a fundamental ambiguity which is present if we discuss "vector spaces" over $\mathbb{R}$*

[1] There's a joke about the financial sector in here somewhere.

or $\mathbb{C}$ incautiously — that is, are we thinking of them algebraically, as sets endowed with operations, or geometrically, as spaces endowed with topologies?

The stakes of such a question are low until we reach algebraic geometry, where topological ideas need to be reformulated and algebraic ones generally need only to be reinterpreted — we have seen already that $\mathbb{A}^n_{\mathbb{C}}$ geometrically realizes the vector space $\mathbb{C}^n$, but this carries some extra points and other data, and it's less than clear how it relates to $\mathbb{C}^n$ in its capacity as a $\mathbb{C}$-module. We will revisit this idea in detail soon — for now, just be warned that other sources use different terminology, and this different terminology can muddy the conceptual waters if employed carelessly.

To reassure ourselves that this is a *co*tangent module and not a tangent one, we observe that our cotangent vectors pull back (rather than pushing forward) along maps of schemes:

Proposition/Definition 2. *Let $\phi : X \to Y$ be a map of schemes and $x \in X$ a point. Then the pullback map $\phi^\# : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ induces a map $\phi^\# : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ of local rings; if we let $\mathfrak{n} \subset \mathcal{O}_{Y,\phi(x)}$ and $\mathfrak{m} \subset \mathcal{O}_{X,x}$ be the maximal ideals, we can see that this in turn induces a map*

$$\phi^* : \mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$$

on Zariski cotangent modules, which we call the **pullback of cotangent vectors***.*

This is our first serious glimpse of the notion that we can do differential geometry on schemes — we will develop this concept in more detail later. For now, we are mainly interested in the relationship with dimension theory:

Exercise 5. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Show that the images of elements $f_1, \ldots, f_r \in \mathfrak{m}$ under the quotient map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ span $\mathfrak{m}/\mathfrak{m}^2$ as a κ-vector space if and only if $\mathfrak{m} = (f_1, \ldots, f_r)$.*

Conclude using Exercise 3 that $\dim R \leq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

(It is common to prove this result using the more general *Nakayama's Lemma*, which we will discuss later. For now, it is not too difficult to show what we need directly.)

That is, the dimension of the cotangent module as a vector space is always at least the local dimension of the scheme. This makes good sense — at the points where the scheme "looks like a manifold", we should expect equality by analogy to differential geometry, and the examples in Figure 5 suggest that things should become more complicated, resulting in extra cotangent directions, at the others. We can now formalize this distinction:

Definition 6. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. (I.e., $\mathfrak{m} \subset R$ is the maximal ideal and $\kappa = R/\mathfrak{m}$ is the residue field.) Then R is called* **regular** *if* $\dim R = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

Let R be a Noetherian ring. Then R is called **regular** *if, for every prime ideal $\mathfrak{p} \subset R$, $R_\mathfrak{p}$ is regular.*

Let X be a locally Noetherian scheme and $x \in X$ a point. Then X is said to be **regular** *or* **nonsingular** *at x if $\mathcal{O}_{X,x}$ is regular — otherwise, we say that X is* **singular** *at x. X itself is called* **regular** *or* **nonsingular** *if it is thus at every point, and* **singular** *otherwise. (Equivalently: X is nonsingular if and only if it admits an affine open cover by the spectra of regular Noetherian rings.)*

This definition is well-posed because, as one might expect, a Noetherian local ring which is regular in the sense of local rings is also regular in the sense of rings (i.e., all of its localizations at smaller prime ideals are regular). (The proof of this fact is actually rather difficult, and took quite some time to be discovered — it uses homological methods beyond the scope of this course.)

Hence we have a notion of dimension which is more topological, in terms of integral closed subschemes, and a notion which is more differential, in terms of cotangent directions, and we think that a scheme should "look like a manifold" exactly where these coincide, as they would in differential geometry. At singular points, as mentioned, we have "extra directions" — perhaps two or more components come together and we get cotangent information from all of them, perhaps the scheme is locally integral but nevertheless folds in on itself in an interesting way, or perhaps the presence of nonreduced structure introduces extra cotangent directions not visible on the level of the underlying space.

Since regularity is defined only for locally Noetherian schemes, which have the property that every closed subscheme contains a closed point, the fact that regular local rings are regular rings implies that regularity can be checked on closed points. In particular, we can verify:

Proposition 6. *Let k be a field and $n \geq 0$ an integer. Then $\mathbb{A}^n_k$ is nonsingular.*

This is exactly what we should hope — after all, affine space is "the algebro-geometric version of Euclidean space", so our notion of "schemes that look like manifolds" should naturally include it.

Exercise 6. *Identify, with proof, all singular points of the following schemes:* $\operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^2)$*,* $\operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^2(x+1))$*,* $\operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^3)$*,* $\operatorname{Spec}\mathbb{C}[x,y,z]/(xz,yz)$*, and* $\operatorname{Spec}\mathbb{Z}$*.*

(It may help to draw pictures!)

As we have alluded to, there is some relationship between nonreducedness and singularity:

Exercise 7. *Show that the "fat line"* $\operatorname{Spec}\mathbb{C}[x,y]/(y^2)$ *is singular at every point (including the generic one).*

More generally, we should expect by the result of Exercise 5 that the presence of nilpotents in the local ring will automatically make a point singular, since these result in extra cotangent directions which cannot be reflected in the dimension of the underlying space:

Exercise 8. *Show that every nonsingular scheme is reduced.*

In particular, (non)singularity is not a purely topological notion, unlike dimension!

Be sure not to lose track, in all of this, of the fact that our definition of singular and nonsingular schemes is applicable only for *locally Noetherian* schemes — to see why this limitation is necessary, and as a general reminder of the gruesomeness of the non-Noetherian setting, we conclude with the following exercise:

Exercise 9. *Let $R = \mathbb{C}[x, x^{1/2}, x^{1/3}, x^{1/4}, \ldots]_{(x, x^{1/2}, x^{1/3}, x^{1/4}, \ldots)}$ be the local ring of Exercise 2. Compute the Zariski cotangent module of* $\operatorname{Spec} R$ *at its closed point.*