The Geometry of Rings and Schemes
Lecture 8: Quasicoherent Sheaves II

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Last lecture, we introduced the notion of a *quasicoherent sheaf* on a scheme — that is, a sheaf which is "affine-locally a module" (or, in the case of sheaves of algebras, "affine-locally an algebra"). Such objects are partially geometric, in that they are defined with respect to the topology of the schemes they live on, and partially algebraic, in that the local module/algebra structures are not themselves expressed in any particularly geometric fashion.

In the case of sheaves of algebras, we saw that we could create a corresponding purely geometric object — a quasicoherent algebra sheaf is given over each affine open by an algebra, which is to say a ring map, and to produce a global object we simply glued the corresponding maps of ring spectra together to produce a single map of schemes, the domain of which we called the *relative spectrum* of the sheaf. We will now do the same for sheaves of modules — we have already seen how to realize a module over a ring as a linear fiber space, so it remains to create a global object by patching these together.

1 Quasicoherent Sheaves as Linear Fiber Spaces

To begin, we generalize our notion of a symmetric algebra from modules to sheaves:

Definition 1. *Let X be a scheme and $\mathcal{F}$ a quasicoherent sheaf of $\mathcal{O}_X$-modules. The* **quasicoherent algebra sheaf** *of $\mathcal{F}$ is the quasicoherent sheaf of $\mathcal{O}_X$-algebras defined by*

$$\mathrm{Sym}(\mathcal{F}) := \frac{\bigoplus_{\ell=0}^{\infty} \mathcal{F}^{\otimes \ell}}{(a \otimes b - b \otimes a \mid a, b \textit{ sections of } \mathcal{F})},$$

where the tensor products are tensor products of sheaves of $\mathcal{O}_X$-modules, we take $\mathcal{O}_X^{\otimes 0}$ to be $\mathcal{O}_X$, and the multiplication operation is given by $\otimes$. (The denominator should be understood as a sheaf of two-sided ideals of the sheaf of noncommutative rings $\bigoplus_{\ell=0}^{\infty} \mathcal{F}^{\otimes \ell}$.)

Affine-locally, this exactly agrees with our previous definition:

Proposition 1. *Let R be a ring and M an R-module. Then* $\mathrm{Sym}(\tilde{M}) = \widetilde{\mathrm{Sym}(M)}$.

*First draft of the TeX source provided by Márton Beke.

We are now ready to formulate our geometric realization for quasicoherent sheaves:

Definition 2. *Let X be a scheme and $\mathcal{F}$ a quasicoherent sheaf of $\mathcal{O}_X$-modules. Then the* **relative spectrum** *of $\mathcal{F}$ is $\operatorname{Spec}_+ \mathcal{F} := \operatorname{Spec}\operatorname{Sym}\mathcal{F}$, endowed with the structure of a linear fiber space over X by affine-locally using the fiberwise vector space operations we defined in the module case.*

As before, we can see that a map of quasicoherent sheaves functorially induces a map of spectra in the opposite direction.

At this point it is worth pausing to get a sense of the big picture. We've defined two notions of a "relative spectrum", one for sheaves of algebras and one for sheaves of modules. In the algebra case, we have:

$$\left\{\begin{matrix}\text{algebra}\\ \uparrow\\ \text{ring}\end{matrix}\right\} \xhookrightarrow[\text{Spec}]{\sim} \left\{\begin{matrix}\text{quasicoherent algebra sheaf}\\ /\\ \text{scheme}\end{matrix}\right\} \xhookrightarrow{\text{Spec}} \left\{\begin{matrix}\text{scheme}\\ \downarrow\\ \text{scheme}\end{matrix}\right\}$$

That is, we start with a purely algebraic notion, that of an algebra over a ring. By taking the spectrum of the base ring and the quasicoherent algebra sheaf corresponding to the algebra, we can view this as an instance of a more general concept, that of a quasicoherent sheaf of algebras over scheme — that is, we have expanded our perspective by taking a geometric view of the base ring, while still tracking the structure of the algebra itself in mostly algebraic terms. By taking the relative spectrum of such a sheaf of algebras, we can again view this as an instance of a more general concept, that of a map of schemes; this also completes the passage from algebra to geometry.

For modules, the picture is similar:

$$\left\{\begin{matrix}\text{module}\\ /\\ \text{ring}\end{matrix}\right\} \xhookrightarrow[\text{Spec}]{\sim} \left\{\begin{matrix}\text{quasicoherent module sheaf}\\ /\\ \text{scheme}\end{matrix}\right\} \xhookrightarrow{\text{Spec}_+} \left\{\begin{matrix}\text{linear fiber space}\\ \downarrow\\ \text{scheme}\end{matrix}\right\}$$

Again, we start with the algebraic notion of a module over a ring, and generalize this to the setting of sheaves over schemes by reinterpreting the base ring geometrically, while leaving the module structure encoded in algebra. With our new definition of the relative spectrum of a sheaf of modules, we can again take the final step to the purely geometric setting — again, this also gives a generalization of the sheaf-theoretic notion, since not every linear fiber space arises from a quasicoherent sheaf.

Remark 1. *Just as every algebra can be regarded as a module by forgetting the ring structure, every quasicoherent sheaf of algebras can be viewed as a sheaf of modules. However, our two notions of relative spectra do not coincide in this way — the relative spectrum of a sheaf of algebras $\mathcal{A}$, just like the spectrum of a ring, depends on the multiplicative structure, and hence $\operatorname{Spec}\mathcal{A}$ and $\operatorname{Spec}_+\mathcal{A}$ are not the same.*

As an example, let $R = \mathbb{C}$ and $A = \mathbb{C}[x]$; then $\operatorname{Spec}(\tilde{A})$ is simply $\mathbb{A}^1_{\mathbb{C}}$, but $\operatorname{Spec}_+(\tilde{A}) = \operatorname{Spec}\operatorname{Sym}(\tilde{A}) = \mathbb{C}[e_0, e_1, e_2, \ldots]$ since $\mathbb{C}[x] \cong \bigoplus_{n=0}^{\infty} \mathbb{C}e_n$ as $\mathbb{C}$-modules.

As in the case of modules, we can retrieve a quasicoherent sheaf from the corresponding linear fiber space using the notion of a linear form:

Definition 3. *Let X be a scheme and V a linear fiber space over X. Then the* **sheaf of linear forms** *on V is the sheaf $\mathcal{L}(V)$ of $\mathcal{O}_X$-modules defined by $\mathcal{L}(V)(U) := L(V|_U)$ for open subschemes $U \subseteq X$, with the addition and multiplication by sections of $\mathcal{O}_X$ defined as in Lecture 6.*

For a map $\Phi : V \to W$ of linear fiber spaces, we also define the corresponding **pullback map** *of sheaves of $\mathcal{O}_X$-modules $\Phi^* : \mathcal{L}(W) \to \mathcal{L}(V)$ by composition as before.*

Here $V|_U$ denotes the restriction $V \times_X U$, which naturally carries the structure of a linear fiber space over U. Note that, despite the fact that its sections are linear forms on V, $\mathcal{L}(V)$ is itself a sheaf on X, not V.

As in the module case, the pullbacks are (contravariantly) functorial. Now, as expected, we find that the sheaf of linear forms on a spectrum retrieves the original sheaf of modules:

Proposition 2. *Let X be a scheme and $\mathcal{F}$ a quasicoherent sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{L}(\operatorname{Spec}_+ \mathcal{F})$ is quasicoherent and, indeed, naturally isomorphic to $\mathcal{F}$.*

Moreover, $\operatorname{Spec}_+(-)$ and $\mathcal{L}(-)$ together define an anti-equivalence of abelian categories between the category of quasicoherent sheaves on X and a full subcategory of the category of linear fiber spaces over X — that is, not every linear fiber space necessarily arises as the spectrum of a quasicoherent sheaf, but any map between those which do is induced by a map of sheaves, and the correspondence respects kernels, cokernels, etc.

This correspondence also behaves well with respect to our various notions of pullback:

Proposition 3. *Let $\phi : X' \to X$ be a map of schemes and $\mathcal{F}$ a quasicoherent sheaf on X. Then there is a natural isomorphism $X' \times_X \operatorname{Spec}_+(\mathcal{F}) \cong \operatorname{Spec}_+(\phi^*\mathcal{F})$ of linear fiber spaces over X'.*

That is, our conversion back and forth between sheaves of modules and linear fiber spaces commutes with pullback, so we can freely identify sheaves with their spectra even in situations where it is important to work over multiple schemes.

It is important to remember that pullback is not, in general, exact. In particular, if we have a short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

of quasicoherent sheaves on a scheme X, Proposition 2 tells us that the corresponding sequence

$$0 \to \operatorname{Spec}_+ \mathcal{C} \to \operatorname{Spec}_+ \mathcal{B} \to \operatorname{Spec}_+ \mathcal{A} \to 0$$

of linear fiber spaces over X is exact, but the fibers of this sequence over points $x \in X$ need not be exact. (Note that here "0" means the zero linear fiber space over X — i.e., the one whose fiber over each point is the zero vector space. This is simply X itself.) That is, writing $V|_x$ for the fiber $V \times_X x$ for each linear fiber space V, we have that the sequence

$$0 \to \operatorname{Spec}_+ \mathcal{C}|_x \to \operatorname{Spec}_+ \mathcal{B}|_x \to \operatorname{Spec}_+ \mathcal{A}|_x$$

of (possibly infinite-dimensional) affine spaces is exact, but the rightmost map is not necessarily onto; equivalently, if $i : x \hookrightarrow X$ is the inclusion,

$$i^*\mathcal{A} \to i^*\mathcal{B} \to i^*\mathcal{C} \to 0$$

is exact, but the leftmost map may not be an inclusion of sheaves.

Example 1. *Let $R = \mathbb{C}[x,y]$ and $X = \operatorname{Spec} R$. Then we have a short exact sequence*

$$0 \to \widetilde{(x,y)} \to \widetilde{R} \to \widetilde{R/(x,y)} \to 0$$

of quasicoherent sheaves of $\mathcal{O}_X$-modules. By considering presentations of these modules, we can see that the corresponding sequence of linear fiber spaces over X is

$$0 \to \operatorname{Spec} \frac{R[e]}{(xe,ye)} \to \operatorname{Spec} R[e] \to \operatorname{Spec} \frac{R[e_1,e_2]}{(ye_1 - xe_2)} \to 0;$$

we will now examine the fibers of this sequence over the closed points of $X = \mathcal{A}^2_{\mathbb{C}}$.

Consider a closed point other than the origin, corresponding to the maximal ideal $\mathfrak{m} = (x-a, y-b)$ in R for $a, b \in \mathbb{C}$ not both zero. To compute the fibers of our linear fiber spaces over this point, we compute $M \otimes_R R/\mathfrak{m} \cong M/\mathfrak{m}M$ for each of our modules M and then take the spectra of the results as modules over $R/\mathfrak{m} \cong \mathbb{C}$.

In the case of the middle term R of our sequence, we obtain simply $R/\mathfrak{m}$; for the module $R/(x,y)$, the result is $R/((x,y)+\mathfrak{m}) = R/(1) = 0$. On the other hand, for $(x,y) \cong \frac{Re_1 \oplus Re_2}{(ye_1 - xe_2)}$, we find that the result is $\frac{\mathbb{C}e_1 \oplus \mathbb{C}e_2}{(be_1 - ae_2)} \cong \mathbb{C}$. Hence, in this case, the sequence of modules

$$0 \to \frac{\mathbb{C}e_1 \oplus \mathbb{C}e_2}{(be_1 - ae_2)} \xrightarrow{\substack{e_1 \mapsto ae \\ e_2 \mapsto be}} \mathbb{C}e \to 0 \to 0$$

we obtain actually is exact, and hence so is the corresponding sequence

$$\begin{tikzcd}
\operatorname{Spec}_+ 0 \arrow[r] \arrow[d, equal] & \operatorname{Spec}_+ 0 \arrow[r] \arrow[d, equal] & \operatorname{Spec}_+ \mathbb{C}e \arrow[r] \arrow[d, equal] & \operatorname{Spec}_+ \frac{\mathbb{C}e_1 \oplus \mathbb{C}e_2}{(be_1 - ae_2)} \arrow[r] \arrow[d, equal] & \operatorname{Spec}_+ 0 \arrow[d, equal] \\
\operatorname{Spec} \mathbb{C} \arrow[r] \arrow[d, equal] & \operatorname{Spec} \mathbb{C} \arrow[r] \arrow[d, equal] & \operatorname{Spec} \mathbb{C}[e] \arrow[r] \arrow[d, equal] & \operatorname{Spec} \frac{\mathbb{C}[e_1,e_2]}{(be_1 - ae_2)} \arrow[r] \arrow[d, equal] & \operatorname{Spec} \mathbb{C} \arrow[d, equal] \\
\mathbb{A}^0_{\mathbb{C}} \arrow[r] & \mathbb{A}^0_{\mathbb{C}} \arrow[r] & \mathbb{A}^1_{\mathbb{C}} \arrow[r] & \mathbb{A}^1_{\mathbb{C}} \arrow[r] & \mathbb{A}^0_{\mathbb{C}}
\end{tikzcd}$$

of linear fiber spaces over the point $\operatorname{Spec} \mathbb{C}$.

On the other hand, if we consider the origin, cut out by the maximal ideal $\mathfrak{m} = (x,y) \subset R$, we can see that the sequence of $R/\mathfrak{m} \cong \mathbb{C}$-modules we obtain is

$$0 \to \mathbb{C}e_1 \oplus \mathbb{C}e_2 \xrightarrow{\substack{e_1 \mapsto 0 \\ e_2 \mapsto 0}} \mathbb{C}e \xrightarrow{\cong} \mathbb{C}e \to 0;$$

this is exact at both $\mathbb{C}e$-terms, but not at $\mathbb{C}e_1 \oplus \mathbb{C}e_2$. Hence we have only an exact sequence

$$\begin{tikzcd}
\operatorname{Spec}_+ 0 \arrow[r] \arrow[d, equal] & \operatorname{Spec}_+ \mathbb{C}e \arrow[r] \arrow[d, equal] & \operatorname{Spec}_+ \mathbb{C}e \arrow[r] \arrow[d, equal] & \operatorname{Spec}_+(\mathbb{C}e_1 \oplus \mathbb{C}e_2) \arrow[d, equal] \\
\operatorname{Spec} \mathbb{C} \arrow[r] \arrow[d, equal] & \operatorname{Spec} \mathbb{C}[e] \arrow[r] \arrow[d, equal] & \operatorname{Spec} \mathbb{C}[e] \arrow[r] \arrow[d, equal] & \operatorname{Spec} \mathbb{C}[e_1, e_2] \arrow[d, equal] \\
\mathbb{A}^0_{\mathbb{C}} \arrow[r] & \mathbb{A}^1_{\mathbb{C}} \arrow[r] & \mathbb{A}^1_{\mathbb{C}} \arrow[r] & \mathbb{A}^2_{\mathbb{C}}
\end{tikzcd}$$

of linear fiber spaces over $\operatorname{Spec}\mathbb{C}$*, where the map on the right is not onto.*

That is, at the origin, our space $\operatorname{Spec}_+(x,y)$ *fails to be the fiberwise cokernel of the map* $\operatorname{Spec}_+ R/(x,y) \to \operatorname{Spec}_+ R$*; however, it is still the cokernel globally and even locally at the level of stalks (i.e., restrictions of linear fiber spaces over the spectrum of the local ring* $R_{(x,y)}$*). Indeed, we will see shortly that the "fiberwise cokernel" we might expect — that is, a linear fiber space which has a zero-dimensional fiber over the origin in* $\operatorname{Spec} R$ *and one-dimensional fibers over the other closed points — does not exist, at least as the spectrum of a module.*

The following special case of our correspondence between quasicoherent sheaves and linear fiber spaces is particularly important:

Proposition/Definition 1. *Let* X *be a scheme,* $n \geq 0$ *an integer, and* V *a linear fiber space over* X*. We call* V *a* **vector bundle of rank** n *over* X *if, for every point* $x \in X$*, there exists an open neighborhood* $U \ni x$ *such that* $V|_U \cong \mathbb{A}^n_U$ *as linear fiber spaces over* U*. This is true if and only if* $V = \operatorname{Spec}_+ \mathcal{F}$ *for* $\mathcal{F}$ *a* **locally free sheaf of rank** n *on* X *— i.e.,* $\mathcal{F}$ *is a quasicoherent sheaf such that, for every point* $x \in X$*, there exists an open neighborhood* $U \ni x$ *such that* $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n}$*.*

Hence every vector bundle is the spectrum of some locally free sheaf, and every locally free sheaf is the sheaf of linear forms on some vector bundle. This follows essentially by noting the equivalence of the respective local triviality conditions.

For those with a background in differential geometry, this notion of a vector bundle is exactly as should be expected — $\mathbb{A}^n_U$ is the "trivial rank-n vector bundle over U" which plays the role which $U \times \mathbb{R}^n$ would in the setting of smooth real manifolds, and so our local triviality requirement is exactly analogous to the classical one.

Remark 2. *Many algebraic geometers take the opposite convention — that is, that a vector bundle should correspond to its sheaf of sections, rather than its sheaf of linear forms. This differs from our correspondence by a dual.*

Now, as previously mentioned, one possible motivation for developing the idea of linear fiber spaces is that the kernels and cokernels of maps of vector bundles are not themselves vector bundles if the maps do not have constant rank. In the locally Noetherian setting, we can see that spectra of coherent sheaves are exactly what we need to construct these objects:

Exercise 1 (optional but not as hard as it may appear)**.** *Let* X *be a locally Noetherian scheme. Show that the category of coherent sheaves on* X *is identified by our correspondence with the full abelian subcategory of the category of linear fiber spaces over* X *generated by the finite-rank vector bundles.*

(Hint: Coherent sheaves locally admit finite presentations; use this to show that their spectra can be realized as the kernels of maps of finite-rank vector bundles. It will then remain to show that the spectra of coherent sheaves form a full abelian subcategory of the category of linear fiber spaces.)

Hence we see that (the spectra of) coherent sheaves are really a quite mild generalization of finite-rank vector bundles, allowing for the variations in fiber dimension necessary to capture the behavior of maps of non-constant rank but introducing little else in the way of new behavior. However, as is always the case in the world of schemes, we have to be careful about taking statements phrased in terms of fibers or points too literally — for instance, it is *not* true that every coherent sheaf with constant fiber dimension is a vector bundle, as the following example shows:

Example 2. *Let $R = \mathbb{C}[x, y]/(y^2)$ and let M be the ring $\mathbb{C}[x, y]/(y)$, considered as an R-module. Then the fiber of $\operatorname{Spec}_+ M$ over any point of $\operatorname{Spec} R$ is the affine line over the residue field, but $\operatorname{Spec}_+ M$ is not a rank-1 vector bundle, since it is not isomorphic to $\mathbb{A}^1_{\operatorname{Spec} R} \cong \operatorname{Spec} R[e]$ over any open set of $\operatorname{Spec} R$.*

That is, not all of the "changes in fiber dimension" which obstruct the spectrum of a coherent sheaf from being a vector bundle necessarily occur on the level of literal fibers, since we also have to account for behavior over the infinitesimal tufts introduced by the presence of nilpotents.

2 Nakayama's Lemma

By the results of the previous section, we can view quasicoherent sheaves as linear fiber spaces — that is, as generalizations of vector bundles with potentially varying fiber dimensions. However, as alluded to in Example 1, the changes in fiber dimension from point to point are not completely arbitrary, at least in the coherent case. The major constraint on their behavior is encapsulated in the following fact from commutative algebra:

Theorem 1 (Nakayama's Lemma)**.** *Let R be a ring, M a finitely-generated R-module, and $I \subseteq R$ an ideal contained in every maximal ideal. Then $IM = M$ if and only if $M = 0$. In particular, a collection of elements $m_1, \ldots, m_n \in M$ generates M if and only if the images of the elements under the quotient map $M \to M/IM$ generate M/IM.*

To see what this means geometrically, note first by our usual translations between containments of ideals and containments of the corresponding closed subschemes that I being contained in every maximal ideal is equivalent to $\operatorname{Spec} R/I$ containing every closed point of $\operatorname{Spec} R$. Now observe that $IM = M$ if and only $M/IM = 0$, which is to say if and only if the pullback of the sheaf $\tilde{M}$ to the closed subscheme $\operatorname{Spec} R/I$ is zero. This is exactly to say that $\operatorname{Spec}_+ M$ restricts to the rank-zero vector bundle over $\operatorname{Spec} R/I$; the assertion is then that this implies $\operatorname{Spec}_+ M$ is the rank-zero vector bundle everywhere.

By specializing this to the case where R is a local ring and I is the maximal ideal, we obtain the following corollary:

Corollary 1. *Let X be a locally Noetherian scheme, $x \in X$ a point, and $\mathcal{F}$ a coherent sheaf on X. Then, if the fiber $\operatorname{Spec}_+ \mathcal{F}|_x$ is zero-dimensional, there is an open subscheme $U \ni x$ such that $\operatorname{Spec}_+ \mathcal{F}|_U$ is the rank-zero vector bundle $\mathbb{A}^0_U$ over U.*

Proof. Let $\operatorname{Spec} R \ni x$ be an affine open subscheme of X, and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to x, with $i : x \hookrightarrow \operatorname{Spec} R$ the inclusion. Then, since $\mathcal{F}$ is coherent, we have $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$ for some finitely-generated R-module M. By hypothesis, we can see that $\operatorname{Spec}_+ \tilde{M}|_x := (\operatorname{Spec} \operatorname{Sym} \tilde{M})|_x \cong \operatorname{Spec} \operatorname{Sym}(i^* \tilde{M}) := \operatorname{Spec} \operatorname{Sym}(\kappa(\mathfrak{p}) \otimes_R M)$ is the zero-dimensional affine space $\mathbb{A}^0_{\kappa(\mathfrak{p})} \cong \operatorname{Spec} \kappa(\mathfrak{p})$ over the residue field, which is to say exactly that $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong \kappa(\mathfrak{p}) \otimes_R M$ is the zero module over $\kappa(\mathfrak{p})$.

Hence, by applying Nakayama's Lemma to $R_\mathfrak{p}$, $M_\mathfrak{p} \cong R_\mathfrak{p} \otimes_R M$, and the maximal ideal $\mathfrak{p}R_\mathfrak{p}$, we can see that $M_\mathfrak{p} = 0$ as well. The remainder of the proof follows our standard argument for lifting facts about local rings to facts about open neighborhoods (see Lecture 4); we note that $R_\mathfrak{p}$ is obtained from R, and hence $M_\mathfrak{p}$ from M, by adjoining formal inverses to all elements not in $\mathfrak{p}$, and so the fact that $M_\mathfrak{p} = 0$ is equivalent to the assertion that, for every $m \in M$, there exists $f \notin \mathfrak{p}$ with $fm = 0$ and hence $m \in \ker(M \to M_f) \subseteq \ker(M \to M_\mathfrak{p})$. Since M is finitely generated, there exist elements $m_1, \ldots, m_n \in M$ of which every element of M is an R-linear combination; letting $f_1, \ldots, f_n \in R \setminus \mathfrak{p}$ be such that $f_i m_i = 0$ for all $1 \leq i \leq n$, we can see that $M_{f_1 \cdots f_n} = 0$. Thus $\operatorname{Spec} R_{f_1 \cdots f_n}$ is an open neighborhood of x on which $\mathcal{F}$ restricts to zero and hence $\operatorname{Spec}_+ \mathcal{F}$ restricts to the rank-zero vector bundle, proving the result. $\square$

Now we turn our attention to the second statement in Theorem 1. To say that the images of $m_1, \ldots, m_n$ in M/IM generate it amounts to saying that the composition

$$R^{\oplus n} \xrightarrow{[m_1 \ \cdots \ m_n]} M \to M/IM$$

is surjective, and Nakayama's Lemma says that this implies the surjectivity of the first map $R^{\oplus n} \xrightarrow{[m_1 \ \cdots \ m_n]} M$. Noting that taking spectra swaps surjections with injections, we can see that this means the following: If $\operatorname{Spec}_+ \tilde{M}|_{\operatorname{Spec} R/I}$ embeds into a trivial rank-n vector bundle, then $\operatorname{Spec}_+ \tilde{M}$ does as well. In particular, we can again specialize to the case of a local ring and maximal ideal to prove the following:

Corollary 2. *Let X be a locally Noetherian scheme, $x \in X$ a point, and $\mathcal{F}$ a coherent sheaf on X. Let $n := \dim \operatorname{Spec}_+ \mathcal{F}|_x$. Then there exists an open subscheme $U \ni x$ of X such that $\operatorname{Spec}_+ \mathcal{F}|_U$ can be embedded as a closed linear fiber subspace of $\mathbb{A}^n_U$ over U. In particular, the fibers of $\operatorname{Spec}_+ \mathcal{F}|_U$ have dimension at most n.*

Proof. This is similar to the proof of the preceding corollary. Again let $\operatorname{Spec} R \ni x$ be an affine open subscheme of X, and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to x. We again have $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$ for some finitely-generated R-module M by the coherence of $\mathcal{F}$.

By the definition of n and the fact that the fibers of a linear fiber space are affine spaces, we see that $\operatorname{Spec}_+ \mathcal{F}|_x \cong \mathbb{A}^n_{\kappa(\mathfrak{p})}$; the corresponding statement on the level of modules is $\kappa(\mathfrak{p})^{\oplus n} \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. By taking the images of the standard basis elements of $\kappa(\mathfrak{p})^{\oplus n}$ under this isomorphism and lifting them arbitrarily to elements $m_1, \ldots, m_n \in M_\mathfrak{p}$, we obtain a map $R_\mathfrak{p}^{\oplus n} \to M_\mathfrak{p}$ of $R_\mathfrak{p}$-modules; applying Nakayama's Lemma to $R_\mathfrak{p}$, $M_\mathfrak{p}$, $\mathfrak{p}$, and $m_1, \ldots, m_n$ tells us that this map is surjective.

Now note that each m_i can be written as a formal fraction $\frac{m_i'}{f_i}$ for some $m_i' \in M$ and $f_i \in R \setminus \mathfrak{p}$. Since the matrix

$$\begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n \end{bmatrix}$$

is invertible over $R_\mathfrak{p}$, we can suppose $m_i = m_i'$ for all $1 \le i \le n$ without altering our surjectivity hypothesis. After this substitution has been made, our map $R_\mathfrak{p}^{\oplus n} \to M_\mathfrak{p}$ will be the result of tensoring the corresponding map $R^{\oplus n} \to M$ of R-modules by $R_\mathfrak{p}$. Let K be the cokernel of this map, so that $R^{\oplus n} \to M \to K \to 0$ is an exact sequence.

Then we can see by the exactness of localization (or, if one prefers, the right-exactness of tensor products in general) that $K_\mathfrak{p} \cong R_\mathfrak{p} \otimes_R K$ is equal to zero by the surjectivity of the corresponding map over the local ring. Since M is finitely-generated, K is as well, and so we can repeat the reasoning used for the previous corollary to produce an affine patch U containing x over which K vanishes, so that $\widetilde{R^{\oplus n}}|_U \to \tilde{M}|_U \to 0$ is exact. The result follows by taking spectra. $\square$

We can rephrase this last observation in the following language:

Definition 4. *Let T be a topological space and $r : T \to \mathbb{R}$ a map of sets. We say that r is* **upper semicontinuous** *if, for each $a \in \mathbb{R}$, $r^{-1}([a, \infty])$ is a closed subset of T.*

In particular, for convergent sequences $x_i \to x$ in T, the upper semicontinuity of r implies $r(x) \ge \limsup_i r(x_i)$. That is, an upper semicontinuous function is one whose value "jumps up only on closed subsets". Our last corollary to Nakayama's Lemma then gives:

Corollary 3 (upper semicontinuity of fiber dimension)**.** *Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Then the function $x \mapsto \dim \operatorname{Spec}_+ \mathcal{F}|_x$ on the underlying space of X is upper semicontinuous.*

This provides the promised constraint on the point-to-point variations of fiber dimension possible for coherent sheaves.

We conclude with a pair of exercises which illustrate the utility of generic points in module-theoretic inquiries:

Exercise 2. *Let X be a locally Noetherian integral scheme and $\mathcal{F}$ a coherent sheaf on X. Let n be the dimension of the fiber of $\operatorname{Spec}_+ \mathcal{F}$ over the generic point of X. Show that all fibers of $\operatorname{Spec}_+ \mathcal{F}$ have dimension at least n and that there exists an open dense subscheme $U \subseteq X$ such that $\operatorname{Spec}_+ \mathcal{F}|_U$ is a (trivial) rank-n vector bundle.*

As a more concrete example of this phenomenon:

Exercise 3. *Let M be a complex matrix whose entries depend algebraically on a parameter t. (E.g., take*

$$M = \begin{bmatrix} 1 & t^2 & t^{12} \\ t^3 - 47 & t^8 + 4 & t^9 \end{bmatrix}$$

or any other matrix with entries polynomial in t.) Let r be the rank of M as a matrix over the field $\mathbb{C}(t)$. Use the result of the previous exercise to show that, for all but finitely many $a \in \mathbb{C}$, setting $t = a$ in M gives a rank-r matrix over $\mathbb{C}$.

(Hint: Regard M as a module map $\mathbb{C}[t]^{\oplus q} \to \mathbb{C}[t]^{\oplus p}$ and consider its cokernel.)