The Geometry of Rings and Schemes
Lecture 10: Calculus on Schemes II

Alex Hof*

November 19, 2024

Last week, we introduced the notion of sheaves of Kähler differentials, which give our algebro-geometric analogue to the sheaf of differential forms on a manifold (or, more broadly, of sheaves of *relative differential forms*), and the corresponding (relative) tangent schemes, and started to introduce some tools for computing these objects explicitly in cases of interest. This time, we'll introduce some more results simplifying these computations and relate our new objects back to previously-discussed concepts such as the Zariski tangent space.

1 The Zariski Normal Scheme

We begin with a generalization of things we have touched on before. Recall from Lecture 5 our concept of *infinitesimal neighborhoods* of points, which record the derivative information, up to some specified order, of our "functions" at the given point. We now have the machinery to define a version of this with an arbitrary closed subscheme in place of our point:

Definition 1. *Let Y be a scheme and $X \hookrightarrow Y$ a closed subscheme, with $\mathcal{I}$ the corresponding ideal sheaf. Then, for $k \geq 0$ an integer, the* **kth-order infinitesimal neighborhood** *of X in Y is the closed subscheme $V(\mathcal{I}^{k+1}) := \operatorname{Spec}(\mathcal{O}_Y/\mathcal{I}^{k+1})$ of Y cut out by $\mathcal{I}^{k+1}$.*

More generally, if $X \hookrightarrow Y$ is a locally closed embedding, we define the **kth-order infinitesimal neighborhood** *of X in Y to be the kth-order infinitesimal neighborhood of X in U for any open subscheme $U \hookrightarrow Y$ such that X is contained in U as a closed subscheme; this is independent of the chosen U.*

Loosely, we can think of this as capturing "derivative information of orders $\leq k$ in directions normal to X", although the precise meaning can be somewhat subtle in general, particularly given that X itself need not be reduced. Note that, if $X = x \in Y$ is a closed point, this definition coincides with that of the infinitesimal neighborhoods of x in Y given in Lecture 5; more generally, if $y \in Y$ is an arbitrary point, not necessarily closed in Y, the kth-order infinitesimal neighborhood of y in Y in the old sense is the kth-order infinitesimal neighborhood of y in $\operatorname{Spec} \mathcal{O}_{Y,y}$ in the new sense.

*First draft of the TeX source provided by Márton Beke.

Example 1. *Let $Y = \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$ and $X = V(y) = \operatorname{Spec} \mathbb{C}[x,y]/(y) \cong \mathbb{A}^1_{\mathbb{C}}$, so that $\mathcal{I} = \widetilde{(y)}$. For each integer $k \geq 0$, we can see that the kth-order infinitesimal neighborhood of X in Y is $\operatorname{Spec} \mathbb{C}[x,y]/(y^{k+1})$; that is, we get subschemes of the affine plane which are all set-theoretically the x-axis, but which have more and more "tangent fuzz in the y-direction" as we increase k.*

Very concretely, if we let $f = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{C}[x,y]$ be a polynomial, we can see that the restriction of f to the kth-order neighborhood — that is, its image under the quotient map $\mathbb{C}[x,y] \to \mathbb{C}[x,y]/(y^{k+1})$ — remembers exactly the data of the coefficients a_{ij} for $j \leq k$, which gives precisely the same information as the restrictions to the x-axis of the iterated partial derivatives $\frac{\partial^j f}{\partial y^j}$ for $0 \leq j \leq k$.

As before, the $k = 1$ case will be of particular interest — just as we isolated the truly first-order derivative information contained in the first-order infinitesimal neighborhood to define the Zariski cotangent module in Lecture 5 and the corresponding Zariski tangent space in Lecture 6, we define:

Definition 2. *Let Y be a scheme, $i : X \hookrightarrow Y$ a closed subscheme, and $\mathcal{I}$ the corresponding ideal sheaf. Then we define the* **conormal sheaf** *of X in Y to be $\mathcal{I}/\mathcal{I}^2$, considered as a quasicoherent sheaf on X. (This is possible since the $\mathcal{O}_Y$-module action on $\mathcal{I}/\mathcal{I}^2$ factors through the quotient map $\mathcal{O}_Y \to \mathcal{O}_Y/\mathcal{I}$; if you like things to be strictly, formally precise, you can say instead that the conormal sheaf is $i^*(\mathcal{I}/\mathcal{I}^2)$.) Moreover, we call the corresponding linear fiber space $N_{X/Y} := \operatorname{Spec}_+(\mathcal{I}/\mathcal{I}^2)$ over X the* **Zariski normal scheme** *of X in Y. (Again, sticklers may want to throw in a pullback along i somewhere here.)*

More generally, if i is a locally closed embedding, we make the corresponding definitions using an arbitrary open subscheme of Y in which X is closed, as in the case of the infinitesimal neighborhoods.

The Zariski normal scheme need not be a vector bundle, but it plays roughly the role occupied by the *normal bundle* in the differential-geometric context, albeit with some added wrinkles. We can begin to make this idea precise using...

2 The Relative (Co)normal Sequence

Last week, we introduced the relative (co)tangent sequence, which, for a map $\Phi : X \to Y$ of schemes over Z, identified $T_{X/Y}$ with the kernel of the relative differential $D_Z\Phi$. In the case where Φ is a closed inclusion, intuition from differential geometry leads us to expect that its differential should be injective. This turns out to be true:

Proposition 1. *Let Y be a scheme and $X \hookrightarrow Y$ a closed subscheme. Then $T_{X/Y} = 0$.*

Proof. It is equivalent to show that $\Omega_{X/Y} = 0$. Indeed, by working affine-locally, we can reduce to showing that, for any surjection $\phi : S \to T$ of rings, $\Omega_{T/S} = 0$. Let $R = \mathbb{Z}$ and observe that S and T are naturally $\mathbb{Z}$-algebras, with ϕ a $\mathbb{Z}$-algebra map.

Then, considering the ring maps $R \to S \to T$, we have the exactness of the corresponding relative cotangent sequence $T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to \Omega_{T/S} \to 0$; that is, $\Omega_{T/S}$ is the cokernel of the natural map $T \otimes_S \Omega_{S/R} \to \Omega_{T/R}$ and hence it suffices to show that this map is surjective.

In our original construction of the Kähler differentials, $\Omega_{T/R}$ was realized as a quotient of $\bigoplus_{g\in T} T dg$, and similarly $T \otimes_S \Omega_{S/R}$ can be presented as a quotient of $\bigoplus_{f\in S} T df$ (using the fact that the tensor product is right exact and so respects presentations). Moreover, we can see that the natural map $T \otimes_S \Omega_{S/R} \to \Omega_{T/R}$ is, in fact, the one induced on the quotients by the map $\bigoplus_{f\in S} T df \to \bigoplus_{g\in T} T dg$ taking df to $d\phi(f)$; surjectivity now follows by the surjectivity of ϕ. $\square$

In particular, the relative differential of a closed inclusion over *any* base scheme Z will be injective, as expected.

In general circumstances, the relative cotangent sequence $\Phi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$ can be extended to a long exact sequence by adding additional terms on the left — these are known as the *André-Quillen homology groups*. In general, their construction is rather involved, and we will not discuss it in detail, but in the case where Φ is a closed inclusion the first of them ends up being manageable:

Proposition 2 (relative (co)normal sequence)**.** *Let $R \to S$ be ring maps, $I \subseteq S$ an ideal, and $T := S/I$. The map $d : I \to \Omega_{S/R}$ given by restricting the universal derivation $d : S \to \Omega_{S/R}$, when composed with the quotient map $\Omega_{S/R} \to T \otimes_S \Omega_{S/R} \cong \Omega_{S/R}/I\Omega_{S/R}$, factors through to a map $I/I^2 \to T \otimes_S \Omega_{S/R}$ by the Leibniz rule. Then the sequence*

$$I/I^2 \to T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to 0$$

given by applying Proposition 1 to the relative cotangent sequence and extending on the left by our new map is exact.

As such, if $X \overset{i}{\hookrightarrow} Y \to Z$ are maps of schemes with i the closed inclusion corresponding to an ideal sheaf $\mathcal{I}$ on Y, then there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$$

of quasicoherent sheaves on X given affine-locally by the one above and hence a corresponding exact sequence

$$0 \to T_{X/Z} \to i^*T_{Y/Z} \to N_{X/Y}$$

of linear fiber spaces over X.

This is close to what we'd see in differential geometry, where the normal bundle of a closed subscheme is realized as the cokernel of the differential of the inclusion map — here, however, we do not have surjectivity on the right in general. In part, this is due to dependence on the base scheme Z — if we take $Z = Y$ and $Y \to Z$ the identity map, for example, then $i^*T_{Y/Z}$ will already be zero regardless of the chosen X. However, even with a well-chosen base scheme, surjectivity is not guaranteed — inclusions of closed subschemes exhibit a much broader range of possible local behaviors than inclusions of closed submanifolds, and the theory is correspondingly more complicated. Nevertheless, the differential-geometric picture provides a valuable starting point for developing intuition on the (co)normal sequence.

The following corollary simplifies our computation of relative tangent schemes for locally finite-type maps:

Corollary 1. *Let R be a ring and $n, r \geq 0$ integers. Set $S := R[x_1, \ldots, x_n]$ and, for any $f_1, \ldots, f_r \in S$, $I := (f_1, \ldots, f_r)$ and $T := S/I$. Then*

$$\Omega_{T/R} \cong \frac{\bigoplus_{i=1}^n T dx_i}{(df_1, \ldots, df_r)},$$

where $df_j = \frac{\partial f_j}{\partial x_1} dx_1 + \cdots + \frac{\partial f_j}{\partial x_n} dx_n$ is the image of f_j under the composition of the universal derivation $d : S \to \Omega_{S/R} \cong \bigoplus_{i=1}^n S dx_i$ with the natural map $\Omega_{S/R} \to T \otimes_S \Omega_{S/R}$ for each $1 \leq j \leq r$.

Consequently,

$$T_{\operatorname{Spec} T/\operatorname{Spec} R} \cong \operatorname{Spec} \frac{R[x_1, \ldots, x_n, dx_1, \ldots, dx_n]}{(f_1, \ldots, f_r, df_1, \ldots, df_r)}.$$

Proof. The identification $\Omega_{S/R} \cong \bigoplus_{i=1}^n S dx_i \cong S^{\oplus n}$ was one of our propositions from last week, and it follows from standard properties of the tensor product that $T \otimes_S \Omega_{S/R} \cong \bigoplus_{i=1}^n T dx_i \cong T^{\oplus n}$.

The fact that $f_1, \ldots, f_r$ generate I can be expressed as the surjectivity of the map $S^{\oplus r} \to I$ taking the jth standard basis element to f_j for each $1 \leq j \leq r$. As such, the composition $R^{\oplus r} \twoheadrightarrow I \twoheadrightarrow I/I^2$ is also surjective, and so, if we further compose this with the natural map $I/I^2 \to T \otimes_S \Omega_{S/R}$ from the relative conormal sequence, we can see that the resulting map $S^{\oplus r} \to T \otimes_S \Omega_{S/R}$ has the same image as $I/I^2 \to T \otimes_S \Omega_{S/R}$, as does the induced map $T^{\oplus r} \to T \otimes_S \Omega_{S/R}$. Therefore, the sequence

$$T^{\oplus r} \to \bigoplus_{i=1}^n T dx_i \to \Omega_{T/R} \to 0$$

is exact, and so the result follows. □

This lets us, for example, simplify some of our computations from last week:

Example 2. *Let $Z = \operatorname{Spec} \mathbb{C}$, $Y = \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, and $X = V(xy-1) = \operatorname{Spec} \mathbb{C}[x, y]/(xy-1)$.*

Then, since $d(xy-1) = d(xy) - d(1) = ydx + xdy - 0 = ydx + xdy$, our result tells us immediately that

$$T_{X/Z} = \operatorname{Spec} \frac{\mathbb{C}[x, y, dx, dy]}{(xy-1, ydx + xdy)} = \operatorname{Spec} \frac{\mathbb{C}[x, y, dx, dy]}{(xy-1, dy + \frac{1}{x^2} dx)},$$

as we computed last week.

We can also use the corollary to verify that the tangent scheme of our old example of a line meeting a plane is as expected:

Example 3. *Let $Z = \operatorname{Spec} \mathbb{C}$ and $Y = \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y, z]$, and consider $X = V(xz, yz) = \operatorname{Spec} \mathbb{C}[x, y, z]/(xz, yz) =: \operatorname{Spec} A$. Then*

$$T_{X/Z} = \operatorname{Spec} \frac{\mathbb{C}[x, y, z, dx, dy, dz]}{(xz, yz, zdx + xdz, zdy + ydz)}$$

by Corollary 1.

We now verify that the restrictions of $T_{X/Z}$ over the parts of X away from the singular point are as expected. For the vertical line, which is given away from the singular point by $\operatorname{Spec} A_z \cong \mathbb{C}[z]_z$, we find that $T_{X/Z}|_{\operatorname{Spec} A_z} \cong \operatorname{Spec} \frac{\mathbb{C}[x,y,z,dx,dy,dz]_z}{(xz,yz,zdx+xdz,zdy+ydz)}$, which is equal to $\operatorname{Spec} \frac{\mathbb{C}[x,y,z,dx,dy,dz]_z}{(x,y,dx+\frac{x}{z}dz,dy+\frac{y}{z}dz)} = \operatorname{Spec} \frac{\mathbb{C}[x,y,z,dx,dy,dz]_z}{(x,y,dx,dy)} \cong \operatorname{Spec} \mathbb{C}[z,dz]_z$; that is, on this subscheme $T_{X/Z} \hookrightarrow T_{Y/Z}$ is given over each point as the vertical line through the origin in $\operatorname{Spec} \mathbb{C}[dx,dy,dz]$, as we would expect for the tangent line to the z-axis.

The part of the plane in X away from the x-axis is given by $\operatorname{Spec} A_y \cong \mathbb{C}[x,y]_y$, and we can see similarly that $T_{X/Z}|_{\operatorname{Spec} A_y} \cong \operatorname{Spec} \frac{\mathbb{C}[x,y,z,dx,dy,dz]_y}{(xz,yz,zdx+xdz,zdy+ydz)} = \operatorname{Spec} \frac{\mathbb{C}[x,y,z,dx,dy,dz]_y}{(z,dz)} \cong \operatorname{Spec} \mathbb{C}[x,y,dx,dy]_y$ is exactly the expected tangent bundle for (an open subcheme of) the xy-plane. The calculation for the complement of the y-axis is entirely analogous.

3 Smoothness

The relative (co)normal sequence links our tangent schemes and Zariski normal schemes to closed subschemes, which themselves generalize the Zariski tangent space — this suggests a relationship between our new and old notions of "tangent space", which we will now expand upon.

For a sequence $X \to Y \to Z$ of maps of schemes, we have seen in Propositions 1 and 2 that $X \to Y$ being a closed inclusion zeroes out a term in the relative (co)tangent sequence while also allowing us to easily extend the sequence by a term on the other side. We now note that this can be taken a step further if $X \to Z$ is also a closed inclusion:

Proposition 3. *Let $R \to S \to T$ be ring maps such that $R \to T$ is surjective (and hence so is $S \to T$), and let $J \subseteq R$ and $I \subseteq S$ be the ideals such that $T \cong R/J \cong S/I$. The map $R \to S$ induces a map $J \to I$, which factors through to a map $J/J^2 \to I/I^2$. The following sequence is then exact:*

$$J/J^2 \to I/I^2 \to T \otimes_S \Omega_{S/R} \to 0$$

As such, if $X \overset{i}{\hookrightarrow} Y \overset{j}{\to} Z$ are maps of schemes such that i and $j \circ i$ are closed inclusions with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$ respectively, we have an exact sequence

$$\mathcal{J}/\mathcal{J}^2 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{Y/Z} \to 0$$

of quasicoherent sheaves on X and a corresponding exact sequence

$$0 \to i^*T_{Y/Z} \to N_{X/Y} \to N_{X/Z}$$

of linear fiber spaces over X.

Proof. Somewhat involved; see Tag 065V of the Stacks Project. □

Although the condition that both i and $j \circ i$ may seem somewhat contrived at first blush, it is useful in many cases of interest — for example, if we have a map of schemes admitting a section, as in the case of the zero section of a linear fiber space, we can get results in the following vein:

Exercise 1. *Let Z be a scheme, $\mathcal{F}$ a quasicoherent sheaf of $\mathcal{O}_Z$-modules, $Y := \operatorname{Spec}_+ \mathcal{F}$ the corresponding linear fiber space over Z, and $X \xhookrightarrow{i} Y$ the closed subscheme given by the zero section of the projection $Y \to Z$. Use Proposition 3 to show that $i^*T_{Y/Z} \cong Y$ as linear fiber spaces over $X \cong Z$ — that is, since Y is a "vector space" (i.e., affine space) fiberwise over Z, its fiberwise tangent spaces at the zero section are simply affine spaces of the same dimensions, fitting together in the same way.*

For our purposes, Proposition 3 is mainly of interest for the following corollary:

Corollary 2. *Let k be a field, X a k-scheme, and $x \in X$ a k-valued point — that is, one such that $\kappa(x) = k$. Then the fiber $T_{X/\operatorname{Spec} k}|_x$ over x of the tangent scheme to X over $\operatorname{Spec} k$ is exactly the Zariski tangent space to X at x.*

Proof. Note that, as a k-valued point, x must be closed in X — so see why, take any affine open neighborhood and observe that the corresponding ring map must be surjective.

We now apply our proposition with the maps $\operatorname{Spec} k = x \hookrightarrow X \to \operatorname{Spec} k$ — since $N_{x/\operatorname{Spec} k} = N_{\operatorname{Spec} k/\operatorname{Spec} k} = 0$, this gives an exact sequence

$$0 \to T_{X/\operatorname{Spec} k}|_x \to N_{x/X} \to 0.$$

The result follows by observing that $N_{x/X}$ is the Zariski tangent space in question. □

For those who want a proof of this fact which does not appeal to the full machinery of Proposition 3, Proposition II.8.7 of Hartshorne will be useful. In the case of non-k-valued points, the situation becomes more complicated (see, e.g., Exercise II.8.1 of Hartshorne), for reasons which we will discuss in more detail shortly — however, in the setting of finite-type schemes over an algebraically closed field which is often of greatest interest to algebraic geometers, it turns out that we can get away without worrying about this for most purposes.

Back in Lecture 5, we distinguished between *regular* or *nonsingular* schemes, which locally look like manifolds in the sense of having correct-dimensional Zariski tangent spaces, and *singular* schemes, which don't. We can now give versions of these notions in the locally finite-type setting which use our new machinery:

Definition 3. *Let k be a field, X a k-scheme, and $n \geq 0$ and integer. We say that X is* **smooth** *of dimension n (over k) if it is locally of finite type (over k), it has pure dimension n (that is, for any Noetherian open subscheme $U \hookrightarrow X$, the irreducible components of U all have dimension exactly n), and $T_{X/\operatorname{Spec} k}$ is a rank-n vector bundle.*

Smoothness will be our "differential-theoretic analogue to regularity" — that is, we should expect smooth k-schemes to behave like smooth manifolds. As a particular example, we have the following refinement to Proposition 2 in this case:

Proposition 4. *Let k be a field and $i : X \hookrightarrow Y$ an inclusion of smooth k-schemes. Then, taking all tangent schemes over $\operatorname{Spec} k$, we have a short exact sequence*

$$0 \to T_X \to i^*T_Y \to N_{X/Y} \to 0$$

given extending the relative normal sequence of Proposition 2. In particular, $N_{X/Y}$ is a vector bundle of rank $\dim Y - \dim X$ over X.

Thus, in these circumstances, the Zariski normal scheme is entirely analogous to the normal bundle.

We now compare smoothness and regularity explicitly:

Proposition 5. *Let k be a field. Then every smooth k-scheme is regular. If k is perfect (that is, of characteristic zero or such that the Frobenius map on k is an automorphism), every regular k-scheme locally of finite type is smooth.*

In particular, in the setting of finite-type $\mathbb{C}$-schemes, smoothness and regularity are the same.

Remark 1. *Many of the challenges here — in particular, the failure of smoothness and regularity to agree in general and the failure of our tangent schemes over fields to give the Zariski tangent spaces over every point — come from the following phenomenon. If $k \hookrightarrow K$ is a field extension, even though the corresponding map $\operatorname{Spec} K \to \operatorname{Spec} k$ is topologically a map of one-point spaces, the relative tangent scheme $T_{\operatorname{Spec} K/\operatorname{Spec} k}$ may be nonzero in general. That is, the field extension encodes some geometric data which shows up at the level of differentials but not of underlying topological spaces.*

For the purposes of this course, we will mostly avoid getting into field theory, but the following consequence to the phenomenon discussed in the preceding remark is of interest:

Proposition 6 (generic smoothness)**.** *Let k be a perfect field and X an integral scheme locally of finite type over k. Then there exists a dense open subscheme $U \hookrightarrow X$ such that U is smooth over k.*

Proof sketch. Let $n := \dim X$ and $\eta \in X$ be the generic point. Field-theoretic considerations then tell us that $\Omega_{\kappa(\eta)/k} \cong \Omega_{X/k}|_\eta$ is an n-dimensional vector space over $\kappa(\eta)$. The result now follows by Exercise 2 of Lecture 8 — note that $\Omega_{X/k}$ is coherent by Corollary 1. $\square$

Of course, in practice we prefer to have some way to actually compute the the loci where smoothness does and doesn't hold. This is given by the following result:

Theorem/Definition 1 (special case of the Jacobian criterion)**.** *Let k be a field, $n, r, d \geq 0$ integers, and $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that the ideal $I = (f_1, \ldots, f_r)$ cuts out a subscheme X of pure dimension d in $\mathbb{A}^n_k$. Then we define the* **Jacobian ideal** *J_X of X to be the ideal of $(n-d) \times (n-d)$ minors of the matrix*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} \end{bmatrix}$$

(with entries considered as elements of $k[x_1, \ldots, x_n]/I$, so that J_X is an ideal of this ring). This is independent of the chosen embedding of X into affine space, and the underlying set of $V(J_X) \hookrightarrow X$ is precisely the locus where X fails to be smooth.

Proof sketch. The result on smoothness partially follows from the proof of Corollary 1 — we realize $\Omega_{X/\operatorname{Spec} k}$ as (the sheaf corresponding to) the cokernel of the map of free modules specified by our matrix above, and observe that J_X will cut out precisely the locus where the matrix's corank is greater than d, corresponding to an increase in the fiber dimension of $T_{X/\operatorname{Spec} k}$. It remains to show that the matrix everywhere has corank at least d — this follows by field-theoretic considerations.

The independence of J_X from choices follows from the theory of *Fitting ideals*, which we will not discuss, at least for the time being. $\square$