The Geometry of Rings and Schemes
Lecture 13: Projectivization II

Alex Hof*

December 10, 2024

We continue our discussion of projectivization and the spaces and maps we obtain thereby.

1 Projective Space

As a concrete example of our general projectivization construction, we introduce the algebro-geometric version of the classical projective spaces discussed last week:

Definition 1. *Let X be a scheme. Then* **projective n-space** *over X is the projectivization $\mathbb{P}^n_X := \mathbb{P}(\mathbb{A}^{n+1}_X)$ of the trivial rank-$(n+1)$ vector bundle over X. If R is a ring, as usual, we may write $\mathbb{P}^n_R$ in place of $\mathbb{P}^n_{\operatorname{Spec} R}$.*

We construct this object explicitly in the simplest case:

Example 1. *Let k be a field and consider $\mathbb{P}^n_k = \mathbb{P}(\mathbb{A}^{n+1}_k) = \operatorname{Proj} k[x_0, \ldots, x_n]$, the "space of lines through the origin in $\mathbb{A}^{n+1}_k$". By definition, this is constructed as a "quotient" of $\mathbb{A}^{n+1}_k \setminus V(x_0, \ldots, x_n)$ in the affine charts given by inverting each positive-degree homogeneous polynomial $h \in k[x_0, \ldots, x_n]$; however, it is in fact enough to consider any subcover of this open cover of the punctured affine space, which is to say that we can restrict our attention to any chosen set of generators of the irrelevant ideal $(x_0, \ldots, x_n)$.*

We will consider the generators $x_0, \ldots, x_n$. For each $0 \leq i \leq n$, let

$$\tilde{U}_i := \operatorname{Spec} k[x_0, \ldots, x_n]_{x_i}$$

be the complement of the coordinate hyperplane $V(x_i)$ in $\mathbb{A}^{n+1}_k$, and observe that these open subschemes indeed form an open cover of $\mathbb{A}^{n+1}_k \setminus V(x_0, \ldots, x_n)$, with transition maps given by the natural isomorphisms $(k[x_0, \ldots, x_n]_{x_i})_{x_j} \cong k[x_0, \ldots, x_n]_{x_i x_j} \cong (k[x_0, \ldots, x_n]_{x_j})_{x_i}$. We now take the quotient of this whole picture under the scaling action by considering the degree-zero parts of all rings involved; specifically, we let

$$U_i := \operatorname{Spec}(k[x_0, \ldots, x_n]_{x_i})_0 = \operatorname{Spec} k\left[\tfrac{x_0}{x_i}, \ldots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \ldots \tfrac{x_n}{x_i}\right] \cong \mathbb{A}^n_k$$

*First draft of the TeX source provided by Márton Beke.

for each $0 \leq i \leq n$ (where the subscript refers to taking the degree-zero part, rather than a further localization), with quotient maps $\tilde{U}_i \to U_i$ induced by the natural ring inclusions $(k[x_0, \ldots, x_n]_{x_i})_0 \hookrightarrow k[x_0, \ldots, x_n]_{x_i}$. Now, the transition maps for the $\tilde{U}_i$ are identifications of $\mathbb{Z}$-graded rings, and hence in particular induce isomorphisms $((k[x_0, \ldots, x_n]_{x_i})_{x_j})_0 \cong (k[x_0, \ldots, x_n]_{x_i x_j})_0 \cong ((k[x_0, \ldots, x_n]_{x_j})_{x_i})_0$ of degree-zero parts; in the local coordinates specified above, these are given by the k-algebra isomorphisms

$$k\left[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_n}{x_i}\right]_{\frac{x_j}{x_i}} \xrightarrow{\sim} k\left[\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots \frac{x_n}{x_j}\right]_{\frac{x_i}{x_j}}$$

taking each $\frac{x_\ell}{x_i}$ to $\left(\frac{x_\ell}{x_j}\right)\left(\frac{x_i}{x_j}\right)^{-1}$. These are our transition maps for the U_i, and so we obtain $\mathbb{P}^n_k$ by gluing along them; the quotient map $\mathbb{A}^{n+1}_k \setminus V(x_0, \ldots, x_n) \to \mathbb{P}^n_k$ is likewise obtained by gluing the maps $\tilde{U}_i \to U_i$.

To reassure yourself about our reduction from the collection of all elements of the irrelevant ideal to the generators, complete the following:

Exercise 1. *Pick your favorite field k, integer $n \geq 0$, and positive-degree homogeneous polynomial $h \in k[x_0, \ldots, x_n]$. Verify that the affine open subscheme $\operatorname{Spec}(k[x_0, \ldots, x_n]_h)_0$ of $\mathbb{P}^n_k$ corresponding to h is covered by its intersections with the affine opens $U_0, \ldots, U_n$ of Example 1.*

In particular, we can retrieve the following scheme, which we first constructed as an exercise in Lecture 2:

Example 2. *Consider $\mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, x_1]$, the space of lines through the origin in the affine plane over $\mathbb{C}$. As noted in Example 1, this scheme is covered by the affine opens $U_0 = \operatorname{Spec} \mathbb{C}[\frac{x_1}{x_0}]$ and $U_1 = \operatorname{Spec} \mathbb{C}[\frac{x_0}{x_1}]$, with the gluing given by the $\mathbb{C}$-algebra isomorphism $\mathbb{C}[\frac{x_1}{x_0}]_{\frac{x_1}{x_0}} \xrightarrow{\sim} \mathbb{C}[\frac{x_0}{x_1}]_{\frac{x_0}{x_1}}$ taking $\frac{x_1}{x_0}$ to $\left(\frac{x_1}{x_1}\right)\left(\frac{x_0}{x_1}\right)^{-1} = \left(\frac{x_0}{x_1}\right)^{-1}$. If we set $x := \frac{x_1}{x_0}$ and $y := \frac{x_0}{x_1}$, we can see that this precisely agrees with our construction from Lecture 2.*

Exercise 2. *Write down charts and transition maps for $\mathbb{P}^2_{\mathbb{C}}$.*

As mentioned, a great deal of work in contemporary algebraic geometry is done in the setting of projective space over a field (often, an algebraically closed one). Of course, just as the interesting things to study in the affine setting turn out to be vanishing loci of collections of polynomials more than, really, the schemes $\mathbb{A}^n_k$ themselves, the real objects of study in projective geometry are closed subschemes of $\mathbb{P}^n_k$. These arise as projectivizations of "unions of lines through the origin in affine spaces" — that is, subschemes of affine spaces preserved by the scaling action:

Proposition/Definition 1. *Let k be a field, $n \geq 0$ an integer, and $I \subseteq k[x_0, \ldots, x_n]$ a homogeneous ideal. Then $\operatorname{Proj} k[x_0, \ldots, x_n]/I$ is naturally a closed subscheme of $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$, with the inclusion induced locally on each affine open $\operatorname{Spec}(k[x_0, \ldots, x_n]_h)_0$ ($h \in k[x_0, \ldots, x_n]$ homogeneous of positive degree) by the degree-zero part of the quotient map $k[x_0, \ldots, x_n]_h \to k[x_0, \ldots, x_n]_h / I k[x_0, \ldots, x_n]_h$. We call this the* **projective vanishing locus** *of I and denote it by $\mathbb{P}V(I)$. (This notation is not completely standard — often this is referred to simply as $V(I)$, with the distinction from the affine vanishing locus left to context, although we will not adapt that convention here.)*

As an easy example, we have the projective vanishing loci of the coordinate hyperplanes:

Example 3. *Let k be a field, $n \geq 0$ an integer, and $0 \leq i \leq n$ another integer, and consider the projective n-space $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$. Then the projective vanishing locus $\mathbb{P}V(x_i) \subseteq \mathbb{P}^n_k$ of the ideal (x_i) is the projectivization $\operatorname{Proj} k[x_0, \ldots, x_n]/(x_i)$ of the ith coordinate hyperplane; by the natural identification of $k[x_0, \ldots, x_n]/(x_i)$ with a polynomial ring in n variables, we can see that this is an isomorphic copy of $\mathbb{P}^{n-1}_k$.*

Of course, more complicated closed subschemes are also possible. Take, for example, the following:

Example 4. *Consider $\mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y, z]$ and $\mathbb{P}^2_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x, y, z]$. The hypersurface $V(z^2 - xy)$ in $\mathbb{A}^3_{\mathbb{C}}$ is a cone (for visualization purposes, note that we have $z = \pm\sqrt{xy}$ or, after the change of coordinates with $x = \tilde{x} - i\tilde{y}$ and $y = \tilde{x} + i\tilde{y}$, $z = \pm\sqrt{\tilde{x}^2 + \tilde{y}^2}$) with an isolated singular point at the origin, and so its projectivization $\mathbb{P}V(z^2 - xy)$ is a nonsingular conic curve in the projective plane $\mathbb{P}^2_{\mathbb{C}}$. To better understand its shape, we consider the intersections with our standard charts.*

In $U_x = \operatorname{Spec} \mathbb{C}[\frac{y}{x}, \frac{z}{x}]$, we find that our curve is given as the spectrum of the degree-zero part of $\mathbb{C}[x, y, z]_x/(z^2 - xy)\mathbb{C}[x, y, z]_x$ and that the degree-zero part of $(z^2 - xy)\mathbb{C}[x, y, z]_x$ is the ideal in $(\mathbb{C}[x, y, z]_x)_0$ generated by $\frac{z^2-xy}{x^2}$. This is to say that our curve's intersection with U_x is given by the closed subscheme $\operatorname{Spec} \mathbb{C}[\frac{y}{x}, \frac{z}{x}]/((\frac{z}{x})^2 - \frac{y}{x})$ of $\operatorname{Spec} \mathbb{C}[\frac{y}{x}, \frac{z}{x}]$, a parabola.

The picture for $U_y = \operatorname{Spec} \mathbb{C}[\frac{x}{y}, \frac{z}{y}]$ is entirely the same once x and y have been interchanged; the result is a parabola $\operatorname{Spec} \mathbb{C}[\frac{x}{y}, \frac{z}{y}]/((\frac{z}{y})^2 - \frac{x}{y})$.

Finally, in $U_z = \operatorname{Spec} \mathbb{C}[\frac{x}{z}, \frac{y}{z}]$, we find that our ideal of $(\mathbb{C}[x, y, z]_z)_0$ is generated by $\frac{z^2-xy}{z^2}$. Hence we obtain the hyperbola $\operatorname{Spec} \mathbb{C}[\frac{x}{z}, \frac{y}{z}]/(1 - \frac{x}{z}\frac{y}{z})$; if we use our change of coordinates from above, we get instead $\operatorname{Spec} \mathbb{C}[\frac{\tilde{x}}{z}, \frac{\tilde{y}}{z}]/(1 - (\frac{\tilde{x}}{z})^2 - (\frac{\tilde{y}}{z})^2)$.

We are now in a position to examine in more detail the idea that projective space is the "space of lines through the origin" in affine space — for simplicity, we work over an algebraically closed field:

Example 5. *Let k be a algebraically closed field and $n \geq 0$ an integer and consider $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$. Take a closed point $p \in \mathbb{P}^n_k$; if we believe that our constructions so far have been the right ones, this should correspond to a line through the origin in $\mathbb{A}^{n+1}_k$, and we will now demonstrate this correspondence explicitly.*

Recalling our open cover $U_0, \ldots, U_n$ of Example 1, we see that we can, without loss of generality, suppose that $p \in U_0 = \operatorname{Spec} k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$. Since k is algebraically closed, the Nullstellensatz guarantees that, for some $a_1, \ldots, a_n \in k$, p is given in U_0 by $V(\frac{x_1}{x_0} - a_1, \ldots, \frac{x_n}{x_0} - a_n)$. Hence, since the quotient map $\tilde{U}_0 \to U_0$ is given by the inclusion of $k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$ into $k[x_0, \ldots, x_n]_{x_0}$ as the degree-zero part, we find that the fiber of the quotient map over this point is $\operatorname{Spec} k[x_0, \ldots, x_n]_{x_0}/(\frac{x_1}{x_0} - a_1, \ldots, \frac{x_n}{x_0} - a_n) = \operatorname{Spec} k[x_0, \ldots, x_n]_{x_0}/(x_1 - a_1 x_0, \ldots, x_n - a_n x_0) = \operatorname{Spec}(k[x_0, \ldots, x_n]/(x_1 - a_1 x_0, \ldots, x_n - a_n x_0))_{x_0}$, a punctured line through the origin in $\mathbb{A}^{n+1}_k$. Is closure, cut out by the kernel of the natural map $k[x_0, \ldots, x_n] \to (k[x_0, \ldots, x_n]/(x_1 - a_1 x_0, \ldots, x_n - a_n x_0))_{x_0}$, is simply the line $V(x_1 - a_1 x_0, \ldots, x_n - a_n x_0)$.

Hence for any closed point in our projective space $\mathbb{P}^n_k$ we obtain a line through the origin in $\mathbb{A}^{n+1}_k$ by taking the closure of the preimage under the quotient map. On the other hand, if *we have a line through the origin, we can see that it will be cut out by some ideal I generated by n linearly independent linear forms, and up to a permutation of coordinates it can thus be written in the form $V(x_1 - a_1x_0, \ldots, x_n - a_nx_0)$ discussed above. By working again in the affine patch U_0, we find that the projectivization $\mathbb{P}V(x_1 - a_1x_0, \ldots, x_n - a_nx_0)$ of the line is a single closed point, given in the local coordinates as $V(\frac{x_1}{x_0} - a_1, \ldots, \frac{x_n}{x_0} - a_n)$. Therefore, as desired, we obtain a bijection between closed points of projective space and lines through the origin in affine space.*

More generally, if we expand our attention to non-closed points and even arbitrary graded rings, we find the following justification for the term "homogeneous spectrum" introduced last week:

Proposition 1. *Let S be an $\mathbb{N}$-graded ring. Then the points of $\operatorname{Proj} S$ correspond to homogeneous prime ideals $\mathfrak{p}$ of S not containing the irrelevant ideal S_+, and the topology is the one generated by open sets of the form $\{\mathfrak{p} \not\supseteq S_+ \text{ homogeneous} \mid \mathfrak{p} \not\ni h\}$ for homogeneous elements h of S. (Equivalently: The closed sets are those of the form $\{\mathfrak{p} \not\supseteq S_+ \text{ homogeneous} \mid \mathfrak{p} \supseteq I\}$ for homogeneous ideals $I \subseteq S$.)*

This is to say that the points of $\operatorname{Proj} S$ can be identified with the points of $\operatorname{Spec} S$ whose closures are closed under the scaling action and not contained in the zero section — that is, those whose closures have well-defined, non-empty projectivizations. (Note, however, that this identification does not give any kind of inclusion of subschemes — for example, contrast the residue fields of the identified closed points and the generic points of the corresponding lines in Example 5.) The distinguished open sets $\{\mathfrak{p} \not\supseteq S_+ \text{ homogeneous} \mid \mathfrak{p} \not\ni h\}$ described here are precisely the underlying sets of the open subschemes $\operatorname{Spec}(S_h)_0$, and the closed sets $\{\mathfrak{p} \not\supseteq S_+ \text{ homogeneous} \mid \mathfrak{p} \supseteq I\}$ are the underlying sets of the closed subschemes $\mathbb{P}(V(I)) = \operatorname{Proj} S/I$.

Now that we have our identification of points in projective space with lines, we can reproduce our classical view of projective n-space as affine n-space plus a "projective $(n-1)$-space at infinity" from last week, which was given by identifying, e.g., the lines through the origin not contained in $V(x_0)$ with their intersections with $V(x_0 - 1)$:

Remark 1. *Let k be a field and $n \geq 0$ an integer, and consider the affine chart $U_0 = \operatorname{Spec}(k[x_0, \ldots, x_n]_{x_0})_0 = \operatorname{Spec} k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$ in $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$, as discussed in Example 1. Then the composed map*

$$(k[x_0, \ldots, x_n]_{x_0})_0 \hookrightarrow k[x_0, \ldots, x_n]_{x_0} \twoheadrightarrow k[x_0, \ldots, x_n]/(x_0 - 1)$$

is an isomorphism; writing the source as $k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$ and the target as $k[x_1, \ldots, x_n]$, we can see that this is the k-algebra map taking $\frac{x_i}{x_0}$ to $\frac{x_i}{1} = x_i$ for each $1 \leq i \leq n$.

By taking spectra, we obtain an isomorphism $V(x_0 - 1) \xrightarrow{\sim} U_0$ which maps each k-valued point of $V(x_0 - 1)$ to the point of U_0 corresponding to the line through the origin it lies on, as in the classical case. Since $V(x_0 - 1) \cong \mathbb{A}^n_k$ and the complement of U_0 in $\mathbb{P}^n_k$ is $\mathbb{P}V(x_0) \cong \mathbb{P}^{n-1}_k$ (as discussed in Example 3), this once again gives us our view of $\mathbb{P}^n_k$ as a "compactification" of $\mathbb{A}^n_k$ by adding a $\mathbb{P}^{n-1}_k$ at the boundary.

More generally, this gives us a way to "compactify" closed subschemes of $\mathbb{A}^n_k$:

Definition 2. *Let k be a field and $n \geq 0$ an integer, and consider a nonzero polynomial $f \in k[x_1, \ldots, x_n]$. Using the typical grading of the polynomial ring, we have a unique decomposition $f = f_0 + \ldots + f_d$ for some integer $d \geq 0$ so that each f_i is homogeneous of degree i and $f_d \neq 0$. The* **homogenization** *of f with respect to x_0 is then the element ${x_0}^d f_0 + {x_0}^{d-1} f_1 + \ldots + x_0 f_{d-1} + f_d$ of the polynomial ring $k[x_0, \ldots, x_n] \cong (k[x_1, \ldots, x_n])[x_0]$; this is a homogeneous element of degree d.*

If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then the **homogenization** *of I with respect to x_0 is the (homogeneous) ideal of $k[x_0, \ldots, x_n]$ generated by the homogenizations of the nonzero elements of I. The projective vanishing of the homogenization of I is called the* **projective closure** *of $V(I)$; as the name suggests, it is the closure of $V(I)$ in $\mathbb{P}^n_k$ if we identify $\operatorname{Spec} k[x_1, \ldots, x_n] = \mathbb{A}^n_k$ with U_0 as in the preceding remark.*

Example 6. *Consider the polynomial $f = x_1 - 1$ in $\mathbb{C}[x_1, x_2]$ and note that $V(f) \cong \mathbb{A}^1_{\mathbb{C}}$ is a line in $\mathbb{A}^2_{\mathbb{C}}$. The homogenization of f with respect to x_0 is $x_1 - x_0$, the vanishing of which defines a plane in $\mathbb{A}^3_{\mathbb{C}}$; this plane's projectivization gives a projective line $\mathbb{P}V(x_1 - x_0) \cong \mathbb{P}^1_{\mathbb{C}}$ in $\mathbb{P}^2_{\mathbb{C}}$, the projective closure of our original affine line.*

Exercise 3. *Pick an integer $n \geq 0$ and a few of your favorite closed subschemes of $\mathbb{A}^n_{\mathbb{C}}$, and compute their projective closures.*

Remark 2. *If k is a field, $n \geq 0$ is an integer, and $f \in k[x_1, \ldots, x_n]$ is a polynomial, we can see that the part of the projective closure of the vanishing of f lying in the hyperplane $\mathbb{P}V(x_0)$ "at infinity" is determined entirely by its highest-degree term. The intuition is that, as we move farther and farther from the origin, the higher-order terms should dominate the lower-order ones, and hence control the shape of the vanishing locus "at the points infinitely far from the origin". More broadly, if I is an ideal of the polynomial ring, we find that the part of the projective closure of $V(I)$ lying in the hyperplane at infinity is determined by the highest-degree terms of elements of I.*

(If I is not a principal ideal, it may not in general be enough to consider the ideal generated by the highest-degree terms of a given set of generators for I, since there may be cancellations among these terms if the generating set is chosen poorly; this problem, loosely speaking, gives an entry point to the theory of *Gröbner bases*, sets of generators for ideals which behave well with respect to the operation of "taking highest-order terms" in some appropriate sense.)

Remark 3. *In general, if X is a scheme and $C \to X$ is a conical fiber space affine over X, then the machinery of Definition 2 can be adapted to give projective closures of subschemes of C in $\mathbb{P}(C \times_X \mathbb{A}^1_X)$. On the level of rings, this is to say that, if S is an $\mathbb{N}$-graded ring, we can formulate a suitable definition of the homogenization of an arbitrary $f \in S$ with respect to x_0, which gives an element of the $\mathbb{N}$-graded ring $S[x_0]$ (considered with the grading such that x_0 has degree 1). Note in particular that the "part at infinity", $\operatorname{Proj} S[x_0]/(x_0)$, will simply be the projectivization $\mathbb{P}(C)$ (for $C = \operatorname{Spec} S$).*

2 Projective Maps

We now zoom out from our study of projective space in particular and return to the examination of projectivizations of conical fiber spaces more generally. The structure maps of projectivizations of (nice) conical fiber spaces are important enough to have their own names:

Definition 3. *Let X be a scheme. A* **projective map** *to X is one which, up to isomorphism, arises as the structure map $\mathbb{P}(C) \to X$ for C a conical fiber space over X which is a closed sub-conical fiber space of $\operatorname{Spec}_+(\mathcal{F})$ for some locally finitely generated quasicoherent sheaf $\mathcal{F}$. (If X is locally Noetherian, this is to say that $\mathcal{F}$ is coherent.)*

In terms of algebras, our definition says that a projective map is one which arises as the structure map $\operatorname{Proj} \mathcal{A} \to X$ for $\mathcal{A}$ an $\mathbb{N}$-graded quasicoherent algebra sheaf on X with $\mathcal{A}_0 = \mathcal{O}_X$ which is locally finitely generated in degree 1.

Remark 4. *Unfortunately, there is some inconsistency in the literature as to the meaning of the term "projective map" — Hartshorne's version, e.g., substitutes a trivial finite-rank vector bundle for the more general $\operatorname{Spec}_+(\mathcal{F})$. This is typically not a big deal, but the definitions do not agree in all circumstances and some results will apply for one but not the other.*

We will not go into very much detail on projective maps for their own sake; however, the following result is worth noting:

Theorem 1. *Every projective map is proper.*

Proof. Since properness is local on the target, we can reduce to the case of the map $\operatorname{Proj} S \to \operatorname{Spec} R$ for R a ring and S an $\mathbb{N}$-graded R-algebra with $S_0 = R$ which is finitely generated in degree 1; write $S = R[x_0, \ldots, x_n]/I$ for I some homogeneous ideal generated in positive degree. We then see by considering affine patches of the domain that our map is of finite type, as required.

Moreover, we can see that it is separated by observing that the $n+1$ affine patches given by the generators $x_0, \ldots, x_n$ of the irrelevant ideal cover $\operatorname{Proj} S$ and, for each $0 \le i, j \le n$, the part of the diagonal morphism $\operatorname{Proj} S \to \operatorname{Proj} S \times_R \operatorname{Proj} S$ lying in the affine open $\operatorname{Spec}(S_{x_i})_0 \times_R \operatorname{Spec}(S_{x_j})_0 \cong \operatorname{Spec}((S_{x_i})_0 \otimes_R (S_{x_j})_0)$ is given by the quotient by the ideal $(\frac{x_0}{x_i} - \frac{x_j}{x_i} \cdot \frac{x_0}{x_j}, \ldots, \frac{x_n}{x_i} - \frac{x_j}{x_i} \cdot \frac{x_n}{x_j})$ (where we understand $\frac{x_i}{x_i}$ and $\frac{x_j}{x_j}$ to be equal to $1 \in R$). Hence the diagonal is indeed a closed embedding.

To verify universal closedness, we will use the valuative criterion; let K be a field and v a valuation on K, and fix maps $\operatorname{Spec} K \to \operatorname{Proj} S$ and $\operatorname{Spec} \mathcal{O}_v \to \operatorname{Spec} R$ making the following diagram commute:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Proj} S \arrow[d] \\ \operatorname{Spec} \mathcal{O}_v \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

It is now enough to show that there is a lift $\operatorname{Spec} \mathcal{O}_v \to \operatorname{Proj} S$ commuting with the given maps.

Since $\operatorname{Spec} K$ is a single point, its image in $\operatorname{Proj} S$ is contained in one of the aforementioned open affine patches; without loss of generality, suppose this image lies in $\operatorname{Spec}(S_{x_0})_0$. Then the map $\operatorname{Spec} K \to \operatorname{Spec}(S_{x_0})_0$ is given by a ring map $(S_{x_0})_0 \to K$, and the existence of our map $\operatorname{Spec}\mathcal{O}_v \to \operatorname{Spec} R$ making the diagram commute precisely guarantees that all elements of R are sent to nonnegatively-valued elements of K. Consider the elements $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$ of $(S_{x_0})_0$ and let $v_1, \ldots, v_n$ be the valuations of their images under the map to K.

If $v_i \geq 0$ for all $1 \leq i \leq n$, then the map $(S_{x_0})_0 \to K$ factors through $\mathcal{O}_v$ and so our map from $\operatorname{Spec}\mathcal{O}_v$ already has a lift within the affine open $\operatorname{Spec}(S_{x_0})_0 \subseteq \operatorname{Proj} S$. If not, suppose without loss of generality that $v_1 \leq v_i$ for all $1 \leq i \leq n$. Then we can see that the valuation of $\frac{x_1}{x_0}$ is negative, and in particular this element is not sent to zero in K. Hence our map $(S_{x_0})_0 \to K$ factors to a map $((S_{x_0})_0)_{\frac{x_1}{x_0}} \to K$ and we have $v((\frac{x_1}{x_0})^{-1}) = -v_1 > 0$ and, for any $2 \leq i \leq n$, $v(\frac{x_i}{x_0}(\frac{x_1}{x_0})^{-1}) = v_i - v_1 \geq 0$. Using the canonical identification $((S_{x_0})_0)_{\frac{x_1}{x_0}} \cong ((S_{x_1})_0)_{\frac{x_0}{x_1}}$, we thus see that in fact the map from $\operatorname{Spec} K$ factors through the intersection of $\operatorname{Spec}(S_{x_0})_0$ and $\operatorname{Spec}(S_{x_1})_0$ and, if we consider the corresponding map $(S_{x_1})_0 \to K$, the elements $\frac{x_0}{x_1}, \ldots, \frac{x_n}{x_1}$ will be sent to nonnegatively-valued elements of K. Hence, as before, the map from $\operatorname{Spec}\mathcal{O}_v$ has a lift within $\operatorname{Spec}(S_{x_1})_0 \subseteq \operatorname{Proj} S$. $\square$

In practice, most proper maps we are interested in will end up being projective; however, as we will discuss in Remark 5, it is possible to construct non-projective proper maps.

The following class of projective maps is often of especial interest, both algebraically and geometrically:

Proposition/Definition 2. *A map $\phi : X \to Y$ of schemes is said to be* **finite** *if it satisfies any of the following equivalent conditions:*

1. *ϕ is affine and, for each affine open $\operatorname{Spec} R \subseteq Y$, if we take $R \to S$ to be the ring map corresponding to the restriction of ϕ over $\operatorname{Spec} R$, the algebra structure on S makes it a finitely-generated R-module. (Note that this is a much stronger condition than being a finitely-generated R-algebra!)*

2. *ϕ is affine and proper.*

3. *ϕ is affine and projective.*

In addition, finite maps have finite fibers (in the sense that the fiber of such a map over any point of the target is set-theoretically a finite union of points) and, if Y is Noetherian, we have the following extra equivalent condition for the finiteness of ϕ:

4. *(Y Noetherian) ϕ is projective and has finite fibers.*

Closed inclusions, being both affine and proper, are an easy class of examples of finite maps; more broadly, one can think loosely that finite maps correspond to proper maps with finite fibers on the topological side, bearing in mind the usual caveats about the relationship between topological and algebro-geometric notions of properness. Note, in particular, that the condition of having finite fibers on its own is not enough to guarantee finiteness in general; open inclusions, for example, are mostly not finite even though all of them fulfill this criterion.

Finite maps are particularly important in the study of closed subschemes of affine space over a field k, where *Noether normalization* guarantees the existence of a finite surjective map from such a subscheme to an appropriate-dimensional affine space; such maps can then be exploited to prove many classical properties of finite-type k-schemes. However, we will not develop this theory in detail.

3 Affine Cones

We now come to a somewhat subtle point in the theory: the relationship between a (relatively affine) conical fiber space and its projectivization. Often — for example, in projective geometry — the projectivization will be seen as the primary object of interest geometrically, but most arguments will still be made in terms of the corresponding graded algebra or sheaf of algebras, since working with a ring is easier than working chartwise with a non-affine scheme. This leads to a disconnect between properties which hold on the projectivization (typically called "geometric") and those which hold on the conical fiber space/graded algebra sheaf itself (typically called "arithmetic") — in Example 4, e.g., we might say we have a projective hypersurface which is "geometrically nonsingular" but "arithmetically singular", since the conic curve itself is smooth, while the algebra $\mathbb{C}[x, y, z]/(z^2 - xy)$ we got it from has a singularity in its spectrum.

In dealing with such issues, the following terminology can be helpful:

Definition 4. *Let X be a scheme and C a conical fiber space over X which is affine over X. Then C is called an* **affine cone** *of $\mathbb{P}(C)$ over X.*

The point is then that "geometric" properties are the properties of $\mathbb{P}(C)$, while "arithmetic" properties are simply the properties of the affine cone C being considered; in particular, they still admit geometric interpretations, despite the terminology.

As the indefinite article in our definition suggests, however, affine cones are not unique:

Example 7. *Let $X = \operatorname{Spec}\mathbb{C}$, and consider $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x, y]$ as a conical fiber space over X in the usual way coming from its vector bundle structure, which corresponds to the standard grading on $\mathbb{C}[x, y]$. Then (y) and (xy, y^2) are both homogeneous ideals, so $C := \operatorname{Spec}\mathbb{C}[x, y]/(y)$ and $C' := \operatorname{Spec}\mathbb{C}[x, y]/(xy, y^2)$ are both closed sub-conical fiber spaces of $\mathbb{A}^2_{\mathbb{C}}$.*

Now, we can see that $C \not\cong C'$, since, e.g., the former is reduced while the latter is not; recall that C' is "C with an extra infinitesimal direction at the origin". However, precisely because the difference is confined to the origin — that is, because the intersections of C and C' with the complement of the origin in $\mathbb{A}^2_{\mathbb{C}}$ are the same — we find that $\mathbb{P}(C)$ and $\mathbb{P}(C')$ are the same, abstractly and even as closed subschemes of $\mathbb{P}^1_{\mathbb{C}} = \mathbb{P}(\mathbb{A}^2_{\mathbb{C}})$.

In this case, one may notice that not all affine cones seem to be created equal; although C and C' both have the same projectivization, C is clearly a more "natural" affine cone for it in the sense that there is nothing extraneous happening at the origin. In general, if we are working inside the projectivization of a fixed larger canonical fiber space, a closed subscheme does admit a kind of "canonical" affine cone inside this ambient space, the one which is *saturated* with respect to the zero section; implicitly, the "arithmetic" properties of closed subschemes of projective space discussed above are typically taken with respect to this saturated cone. For more details on this construction, see, e.g., Section 15.7 of Vakil (after reading Section 4 below).

However, this very much depends on the chosen ambient conical fiber space, and without such a fixed embedding deeper issues will arise. As a first step toward exploring these, we introduce a kind of functoriality of projectivization:

Proposition/Definition 3. *Let X be a scheme, C and D conical fiber spaces affine over X, and $\gamma : \mathbb{A}^1_X \to \mathbb{A}^1_X$ a map of fiberwise monoids-with-zero over X (i.e., γ is given by the $\mathcal{O}_X$-algebra map $\mathcal{O}_X[t] \to \mathcal{O}_X[t]$ taking t to t^n for some $n > 0$). Suppose we have a map $\phi : C \to D$ of X-schemes such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{A}^1_X \times_X C & \xrightarrow{\cdot} & C \\ \downarrow{\scriptstyle \gamma\times\phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{A}^1_X \times_X D & \xrightarrow{\cdot} & D \end{array}$$

(Equivalently: We have a map $\pi_{D*}\mathcal{O}_D \to \pi_{C*}\mathcal{O}_C$ of the corresponding $\mathcal{O}_X$-algebra sheaves such that, for each $d \in \mathbb{N}$, the homogeneous degree-d part $(\pi_{D*}\mathcal{O}_D)_d$ maps into $(\pi_{C*}\mathcal{O}_C)_{nd}$ for the same $n > 0$. Here $\pi_C : C \to X$ and $\pi_D : D \to X$ denote the projection maps.)

Then, if we let $X \cong Z_D \hookrightarrow D$ be the zero section of D and denote by $\phi^* Z_D$ the locus $C \times_D Z_D$ in C mapping into it under ϕ, there is an induced map $\mathbb{P}(\phi) : \mathbb{P}(C) \setminus \mathbb{P}(\phi^* Z_D) \to \mathbb{P}(D)$ on projectivizations which is functorial in the appropriate sense (i.e., the construction respects compositions over the locus where all relevant maps are defined). We call this map the **projectivization** *of ϕ.*

(The role of γ is just to give a bit more flexibility in the kinds of maps we can consider — instead of thinking only of those which are equivariant under our $\mathbb{A}^1_X$-action, we allow also maps equivariant up to a twist by some monoid-with-zero endomorphism of $\mathbb{A}^1_X$.)

Proof sketch. By working affine-locally on X, we can consider only the case of $\mathbb{N}$-graded R-algebras S, T with $S_0 = T_0 = R$ and an R-algebra map $\psi : S \to T$ multiplying all degrees by some $n > 0$. For any homogeneous positive-degree element $h \in S_+$, we have an induced map $S_h \to T_{\psi(h)}$ of $\mathbb{Z}$-graded algebras which again multiplies all degrees by n; in particular, this restricts to a map $(S_h)_0 \to (T_{\psi(h)})_0$ of degree-zero parts, giving a map from the appropriate affine patch $\operatorname{Spec}(T_{\psi(h)})_0$ of $\mathbb{P}(C) = \operatorname{Proj} T$ to the affine patch $\operatorname{Spec}(S_h)_0$ of $\mathbb{P}(D) = \operatorname{Proj} S$. Since the zero section Z_D is cut out in $D = \operatorname{Spec} S$ by the irrelevant ideal S_+ and so $\phi^* Z_D$ is cut out in $C = \operatorname{Spec} T$ by $\phi(S_+)T$, we can see that the affine patches $\operatorname{Spec} T_{\psi(h)}$ for such h give an open cover of $C \setminus \phi^* Z_D$; hence the corresponding affine patches $\operatorname{Spec}(T_{\psi(h)})_0$ collectively form an open cover of $\mathbb{P}(C) \setminus \mathbb{P}(\phi^* Z_D)$ and thus these maps will give us our $\mathbb{P}(\phi)$.

Strictly speaking, we should verify that these maps glue properly across the different affine patches, but we omit this part of the argument. Similarly, we leave the verification of functoriality to the reader, noting only that, technically, we should work in the category whose objects are conical fiber spaces over X and whose morphisms are pairs (γ, ϕ) of the form we are considering; on the level of algebras or sheaves of algebras, this is describable more simply as the category of $\mathbb{N}$-graded algebra sheaves with degree-zero part $\mathcal{O}_X$, with morphisms given by maps multiplying the degree by specified positive integers. $\square$

The inclusions of the projective vanishing loci of Proposition/Definition 1 into projective space can now be subsumed into our construction here; they are simply the maps induced on projectivizations in this way by the corresponding inclusions of sub-conical fiber spaces. Thus, more generally, we can likewise define an appropriate notion of the projective vanishing of any homogeneous ideal in a graded ring or homogeneous ideal sheaf in a graded algebra sheaf.

Our introduction of the twist by γ allows for constructions such as the following:

Definition 5. *Let S be a $\mathbb{Z}$-graded ring and $n > 0$ an integer. Then the nth* **Veronese subring** *of S is the ring $S\{n\} := \bigoplus_{d\in\mathbb{Z}} S_{nd}$ with the $\mathbb{Z}$-grading such that $S\{n\}_d = S_{nd}$ for all $d \in \mathbb{Z}$. We regard $S\{n\}$ as being naturally endowed with the inclusion $S\{n\} \hookrightarrow S$, which multiplies degrees by n.*

Likewise, if X is a scheme, $\mathcal{A}$ is a quasicoherent $\mathbb{Z}$-graded sheaf of $\mathcal{O}_X$-algebras, and $n > 0$ is an integer, we define the nth **Veronese subalgebra sheaf** *of $\mathcal{A}$ to be $\mathcal{A}\{n\} := \bigoplus_{d\in\mathbb{Z}} \mathcal{A}_{nd}$ with the $\mathbb{Z}$-grading $\mathcal{A}\{n\}_d = \mathcal{A}_{nd}$ and the inclusion $\mathcal{A}\{n\} \hookrightarrow \mathcal{A}$.*

If our original grading is, in fact, an $\mathbb{N}$-grading (that is, the negative-degree parts are all zero), the resulting Veronese subobjects will have zero negative-degree parts as well, and so we may also regard them as $\mathbb{N}$-graded objects. In particular, if X is a scheme, $C \xrightarrow{\pi} X$ is a conical fiber space affine over X, and $n > 0$ is again an integer, then $C\{n\} := \operatorname{Spec}((\pi_*\mathcal{O}_C)\{n\})$ is also a conical fiber space over X, which we call the nth **Veronese twist** *of C; we regard this as being endowed with the map $C \to C\{n\}$ induced by the algebra inclusion $(\pi_*\mathcal{O}_C)\{n\} \hookrightarrow \pi_*\mathcal{O}_C$.*

(The curly brace notation and the term "Veronese twist" are both nonstandard, but there do not appear to be widely-accepted alternatives in the literature.)

The geometric intuition for these objects is as follows. For $n > 0$ an integer, consider the quotient map $\mathbb{C}[t] \to \mathbb{C}[t]/(t^n - 1)$. Then the corresponding map $\operatorname{Spec}\mathbb{C}[t]/(t^n - 1) \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$ is the inclusion of the n closed points corresponding to the nth roots of unity in $\mathbb{C}$, and we can see that the monoid structure of $\mathbb{A}^1_{\mathbb{C}}$ induces a group structure on $\operatorname{Spec}\mathbb{C}[t]/(t^n - 1)$ by restriction, so that we can in fact regard it as the algebro-geometric realization of the group of nth roots of unity in $\mathbb{C}$. Note that the inclusion into the affine line factors as $\operatorname{Spec}\mathbb{C}[t]/(t^n - 1) \hookrightarrow (\mathbb{A}^1_{\mathbb{C}})^* \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$; that is, the group of nth roots of unity is a subgroup of the punctured affine line.

Likewise, we can think for a scheme X that the relative spectrum of $\mathcal{O}_X[t]/(t^n - 1)$ gives the "fiberwise group of nth roots of unity". In particular, for S an $\mathbb{N}$-graded ring, the usual $\mathbb{A}^1_{S_0}$-action $\mathbb{A}^1_{S_0} \times_{S_0} \operatorname{Spec} S \to S$ given by the map $S \to S[t]$ taking each $h \in S_d$ to $t^d h$ can be restricted to an action $(\operatorname{Spec} S_0[t]/(t^n - 1)) \times_{S_0} \operatorname{Spec} S \to \operatorname{Spec} S$ of the nth roots of unity simply by taking the composition $S \to S[t] \to S[t]/(t^n - 1)$. We then see that the nth Veronese subring of S is precisely the ring of invariants of S under this restricted group action, since for any $d \in \mathbb{N}$ and $h \in S_{nd}$ we have $h \mapsto t^{nd}h = (t^n)^d h \mapsto 1^d h = h$ under our composition.

Therefore, by applying this observation in affine charts and adopting once more the geometric invariant-theoretic viewpoint that spectra of rings of invariants should give quotients, we find for a conical fiber space C affine over a scheme X that the nth Veronese twist $C\{n\}$ is simply the quotient of C under multiplication (fiberwise over X) by the nth roots of unity, with the natural map $C \to C\{n\}$ being the quotient map.

Example 8. *Consider the affine line $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x]$ as a conical fiber space over $\operatorname{Spec}\mathbb{C}$. Then the quotient map $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}\{2\}$ for the 2nd Veronese twist is given by the inclusion $\mathbb{C}[x^2] \hookrightarrow \mathbb{C}[x]$; since $\mathbb{C}[y] \cong \mathbb{C}[x^2]$, we can regard this as a map from the affine line to itself, which is the algebro-geometric realization of the usual square map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$.*

Example 9. *Now consider the affine plane* $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ *as a conical fiber space over* $\operatorname{Spec} \mathbb{C}$. *Then* $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}\{2\}$ *is given by the inclusion* $\mathbb{C}[x^2, xy, y^2] \hookrightarrow \mathbb{C}[x, y]$; *using the isomorphism* $\mathbb{C}[u, v, w]/(v^2 - uw) \cong \mathbb{C}[x^2, xy, y^2]$ *taking* u *to* x^2, v *to* xy, *and* w *to* y^2, *we see that this map realizes the cone of Example 4 as a quotient of the affine plane (albeit with a change of coordinates). In particular, since we are considering the group of square roots of unity, we can think of this as the quotient by the antipodal map* $x \mapsto -x, y \mapsto -y$.

Through this lens, the modification of the grading can be interpreted as follows. If we were to give each $S\{n\}$ the grading inherited from S, the corresponding $\mathbb{A}^1_{S_0}$-action on $\operatorname{Spec} S\{n\}$ would then be the one induced by the quotient map $\operatorname{Spec} S \to \operatorname{Spec} S\{n\}$. However, the group of nth roots of unity would then, by definition, act trivially, so we would lose nothing by factoring through to the action of the monoid which is the quotient of $\mathbb{A}^1_{S_0}$ by the submonoid of nth roots of unity. Algebraically, this quotient is given by the subring $S[t^n]$ of $S[t]$, which we can see is isomorphic to the polynomial algebra $S[t']$ for $t' = t^n$; in particular, the result of taking the quotient is simply another monoid-with-zero action of the affine line on $\operatorname{Spec} S$, and so we take instead the grading corresponding to this action. This yields the grading convention of Definition 5.

Since the Veronese twists are defined as quotients of a conical fiber space by subgroups of the punctured affine line, the following result is, on some level, to be expected:

Proposition 2. *Let* X *be a scheme,* C *a conical fiber space affine over* X, *and* $n > 0$ *an integer. Then the projectivization of the quotient map* $C \to C\{n\}$ *for the* n*th Veronese twist gives an isomorphism* $\mathbb{P}(C) \xrightarrow{\sim} \mathbb{P}(C\{n\})$.

That is, since we projectivize by throwing away the zero section and modding out by the action of the punctured affine line, already having taken a quotient by the action of a subgroup does not change the final result.

Proof. We work affine-locally; let S be an $\mathbb{N}$-graded ring and take $X = \operatorname{Spec} S_0$ and $C = \operatorname{Spec} S$. Let $\phi : C \to C\{n\}$ denote the quotient map and Z_C and $Z_{C\{n\}}$ the closed subschemes of C and $C\{n\}$ respectively given by the zero sections.

The part $\phi^* Z_{C\{n\}}$ of C mapping into the zero section of $C\{n\}$ is cut out by the ideal $(S\{n\}_+)S$ generated by all homogeneous elements of S with degrees which are positive multiples of n; since every element of S_+ is nilpotent modulo this ideal, we can see that the underlying sets of $\phi^* Z_{C\{n\}}$ and Z_C are the same. Thus $\mathbb{P}(\phi)$ is, indeed, defined on all of $\mathbb{P}(C)$.

To show that it is an isomorphism of schemes, we must argue that, for every positive $d \in \mathbb{N}$ and $h \in S\{n\}_d = S_{nd}$, the induced map $(S\{n\}_h)_0 \to (S_h)_0$ on degree-zero parts is an isomorphism of rings; since these affine patches cover $\mathbb{P}(C)$ and $\mathbb{P}(C\{n\})$, this will demonstrate that $\mathbb{P}(\phi)$ is an isomorphism globally as well. Since localizations and taking degree-zero parts both preserve inclusions, we can see that $(S\{n\}_h)_0 \to (S_h)_0$ is injective. Now observe that each element of $(S_h)_0$ is of the form $\frac{g}{h^k}$ for $k \geq 0$ and $g \in S_{knd}$, since h^k is of degree knd and the degrees of numerator and denominator must cancel for our element to be of degree zero. Hence $g \in S\{n\}_{kd}$, so our element has a preimage $\frac{g}{h^k}$ in $(S\{n\}_h)_0$ and thus the map is surjective as well. The result follows. $\square$

Therefore, to return to the original matter under discussion, we see that there is a substantial potential for nonuniqueness of affine cones even when we set aside the issue of nonreduced behavior at the origin; the projectivization on its own simply does not remember enough information about the original conical fiber space to distinguish it from its Veronese twists, let alone other possible candidates.

Remark 5. *One consequence of this nonuniqueness is that being a projective map is not a local condition on the target. That is, although the axioms for a conical fiber space $C \to X$ can certainly be checked target-locally, it may be the case for open $U, U' \subseteq X$ and conical fiber spaces $C \to U$, $C' \to U'$ that we have an isomorphism $\mathbb{P}(C|_{U\cap U'}) \cong \mathbb{P}(C'|_{U\cap U'})$ over $U \cap U'$ not induced by any isomorphism $C|_{U\cap U'} \cong C'|_{U\cap U'}$ of conical fiber spaces over $U \cap U'$. Hence, when we glue together along the overlap, we obtain a map that is locally given by projectivizations without itself possessing this property.*

Since properness of maps of schemes is a local condition on the target, and projective maps are proper, we can thus see that any map which is target-locally projective but not projective will also be a non-projective proper map; an explicit example was given by Hironaka in his thesis.

4 Graded Modules and the Tautological Bundle

As discussed in the previous section, affine cones are not unique, and projectivization forgets some of the information of the original conical fiber space; however, as we will now see, in nice cases much of this information can be recorded in the form of a particular line bundle on the projectivized space, called the *tautological bundle*. To this end, we also develop some of the theory of quasicoherent sheaves on projectivizations in general:

Definition 6. *Let S be a $\mathbb{Z}$-graded ring. A* **($\mathbb{Z}$-)graded S-module** *is an S-module M with a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$ as abelian groups such that, for each $i, j \in \mathbb{Z}$, $S_i M_j \subseteq M_{i+j}$. For such an M and $n \in \mathbb{Z}$, we define the nth* **(Serre) twist** *of M to be the graded S-module $M(n)$ such that $(M(n))_d = M_{n+d}$ for all $d \in \mathbb{Z}$ and the multiplication by S is induced by the multiplication on M; that is, $M(n)$ is M with the degree of each element lowered by n.*

Similarly, if X is a scheme and $\mathcal{A}$ is an $\mathbb{Z}$-graded quasicoherent sheaf of $\mathcal{O}_X$-algebras, a **($\mathbb{Z}$-)graded (quasicoherent) sheaf of $\mathcal{A}$-modules** *is a quasicoherent sheaf $\mathcal{F}$ on X with an $\mathcal{A}$-module structure and a decomposition $\mathcal{F} = \bigoplus_{d\in\mathbb{Z}} \mathcal{F}_d$ as sheaves of abelian groups such that, for each $i, j \in \mathbb{Z}$, $\mathcal{A}_i \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$. We define the nth* **Serre twist** *$\mathcal{F}(n)$ of $\mathcal{F}$ for $n \in \mathbb{Z}$ by $(\mathcal{F}(n))_d = \mathcal{F}_{n+d}$ for $d \in \mathbb{Z}$, as before.*

(For simplicity, we are dealing only with $\mathbb{Z}$-graded modules over $\mathbb{Z}$-graded rings — the definitions for $\mathbb{Z}$-graded modules over $\mathbb{N}$-graded rings and $\mathbb{N}$-graded modules over $\mathbb{N}$-graded rings are, however, exactly the special cases you would expect.)

Geometrically, the point is that, for X a scheme, $\mathcal{A}$ a $\mathbb{Z}$-graded quasicoherent algebra sheaf on X, and $\mathcal{F}$ a graded sheaf of $\mathcal{A}$-modules, we can regard $\mathcal{F}$ as a quasicoherent sheaf on $\operatorname{Spec} \mathcal{A}$ and take its spectrum $\operatorname{Spec}_+ \mathcal{F}$ to get a linear fiber space over $\operatorname{Spec} \mathcal{A}$ with an action of $(\mathbb{A}^1_X)^*$ which is compatible with the one on $\operatorname{Spec} \mathcal{A}$ (note that this is not the usual fiberwise action coming from the vector space structure). That is, over each $(\mathbb{A}^1_X)^*$-orbit in

$\operatorname{Spec}\mathcal{A}$, $\operatorname{Spec}_+\mathcal{F}$ restricts to a vector bundle which is invariant under pullbacks along the automorphisms of the orbit induced by the action.

If, in particular, $\mathcal{A}$ is $\mathbb{N}$-graded with $\mathcal{A}_0 = \mathcal{O}_X$, we then see that the restriction of $\mathcal{F}$ over the complement of the image Z of the zero section gives rise to a linear fiber space which is constant along the fibers of the quotient map $\operatorname{Spec}\mathcal{A} \setminus Z \to \operatorname{Proj}\mathcal{A}$. As we should thus expect, this yields a well-defined quotient linear fiber space over the projectivization $\operatorname{Proj}\mathcal{A}$:

Proposition/Definition 4. *Let S be an $\mathbb{N}$-graded ring and M a $\mathbb{Z}$-graded S-module. Then the assignments $\mathcal{P}(\tilde{M})|_{\operatorname{Spec}(S_h)_0} := \widetilde{(M_h)_0}$ on the affine open patches of $\operatorname{Proj} S$ given by $\operatorname{Spec}(S_h)_0$ for $h \in S_+$ homogeneous, together with the natural restriction maps on the overlaps, give rise to a well-defined quasicoherent sheaf $\mathcal{P}(\tilde{M})$ on $\operatorname{Proj} S$, which we call the* **induced sheaf** *of M (or of $\tilde{M}$) on $\operatorname{Proj} S$.*

Likewise, let X be a scheme and $C \xrightarrow{\pi} X$ a conical fiber space affine over X, with $\mathcal{A} = \pi_*\mathcal{O}_C$ the corresponding algebra sheaf. Then, for each $\mathbb{Z}$-graded quasicoherent sheaf $\mathcal{F}$ of $\mathcal{A}$-modules, the induced sheaves $\mathcal{P}(\mathcal{F}|_{\pi^{-1}(\operatorname{Spec} R)})$ defined over affine patches $\operatorname{Spec} R \hookrightarrow X$ naturally glue together into a well-defined quasicoherent sheaf $\mathcal{P}(\mathcal{F})$ on $\mathbb{P}(C)$, called the **induced sheaf** *of $\mathcal{F}$ on $\mathbb{P}(C)$. This $\mathcal{P}$ is then an additive functor from the category of $\mathbb{Z}$-graded quasicoherent sheaves of $\mathcal{A}$-modules to the category of of quasicoherent sheaves of $\mathcal{O}_{\mathbb{P}(C)}$-modules.*

(The notation $\mathcal{P}(\tilde{M})$ is nonstandard — typically, people refer to this sheaf simply as $\tilde{M}$, relying on the fact that they are working in the graded context to distinguish it from the usual induced sheaf $\tilde{M}$ on $\operatorname{Spec} S$. Since we are taking a perspective in which it is important to be able to talk about the affine cone $\operatorname{Spec} S$ as well as the homogeneous spectrum $\operatorname{Proj} S$, we adopt the stated convention as a way of distinguishing between these objects.)

Proof sketch. We note first that each M_h does indeed have a natural grading such that $\deg \frac{m}{h^k} = \deg m - k \deg h$ for each $m \in M$, making M_h a graded S_h-module and, in particular, inducing a $(S_h)_0$-module structure on $(M_h)_0$. The transition maps between $\operatorname{Spec}(S_h)_0$ and $\operatorname{Spec}(S_{h'})_0$ in $\operatorname{Proj} S$ are given by the natural identifications

$$((S_h)_0)_{\frac{h'^{\deg h}}{h^{\deg h'}}} \cong (S_{hh'})_0 \cong ((S_{h'})_0)_{\frac{h^{\deg h'}}{h'^{\deg h}}},$$

and we can see that these induce corresponding gluing isomorphisms

$$((M_h)_0)_{\frac{h'^{\deg h}}{h^{\deg h'}}} \cong (M_{hh'})_0 \cong ((M_{h'})_0)_{\frac{h^{\deg h'}}{h'^{\deg h}}},$$

which we use to construct our $\mathcal{P}(\tilde{M})$. We omit verification of the details. Likewise, the gluing-together over the scheme X rests on natural identifications between these constructions induced by the gluing isomorphisms between affine patches $\operatorname{Spec} R$ of X, which we will not trouble ourselves to write out in full.

For the additive functoriality, it is enough to observe that all of the affine-patch-wise operations used are additively functorial, as are the gluing operations used to combine everything into the final sheaf. We continue to neglect the details; the point is that maps of modules M induce maps of sheaves $\mathcal{P}(\tilde{M})$ and that this process respects the abelian group structure on module maps, as one would expect, and that the analogous statements hold for the assignment $\mathcal{F} \mapsto \mathcal{P}(\mathcal{F})$. $\square$

Remark 6. *It is natural to wonder whether every quasicoherent sheaf $\mathcal{G}$ on $\mathbb{P}(C)$ arises in this way. Certainly, if we let Z be the image of the zero section and $q : C \setminus Z \to \mathbb{P}(C)$ be the quotient, we can see that any graded sheaf inducing $\mathcal{G}$ should restrict to the pullback sheaf $q^*\mathcal{G}$ on $C \setminus Z$; here the grading corresponds to constancy along the $(\mathbb{A}^1_X)^*$-orbits, which are precisely the fibers of q, and so we see that $q^*\mathcal{G}$ is naturally $\mathbb{Z}$-graded simply by virtue of being a pullback. (Algebraically: If we are working affine-locally on X, so that $C = \operatorname{Spec} S$ for some $\mathbb{N}$-graded ring S, we can see explicitly for each homogeneous $h \in S_+$ with $\mathcal{G}|_{\operatorname{Spec}(S_h)_0} =: \tilde{G}$ that $i^*\mathcal{G}$ is given on the corresponding affine open $\operatorname{Spec} S_h$ of $C \setminus Z$ by the graded module $G \otimes_{(S_h)_0} S_h \cong \bigoplus_{d\in\mathbb{Z}} G \otimes_{(S_h)_0} (S_h)_d$.)*

It then remains to ask whether there is a quasicoherent sheaf on C extending $q^*\mathcal{G}$. If $i : C\setminus Z \hookrightarrow C$ is the inclusion, one natural candidate is of course the pushforward sheaf $i_*q^*\mathcal{G}$; however, we must recall from Lecture 7 that pushforward does not preserve quasicoherence in general, and we need extra conditions such as quasicompactness and quasiseparatedness for this to occur. As an open inclusion, i will be quasiseparated in general, but to obtain quasicompactness we must require that $\mathcal{A}$ be locally finitely generated as an $\mathcal{O}_X$-algebra sheaf.

Hence, under nice conditions — such as when $\mathcal{A}$ is locally finitely generated in degree 1, so that $\mathbb{P}(C) \to X$ is projective — we find that $\pi_*i_*q^*\mathcal{G}$ is a $\mathbb{Z}$-graded quasicoherent sheaf of $\mathcal{A}$-modules on X (note that π is qcqs by virtue of being affine, guaranteeing the quasicoherence) with $\mathcal{P}(\pi_*i_*q^*\mathcal{G}) = \mathcal{G}$.

For a more detailed treatment of this construction, see Section 15.7 of Vakil, as mentioned above; to understand why Vakil's approach is the same as ours, see Remark 7 below.

We are now ready to define the tautological bundle. The concept is that, if our affine cone C is a sub-conical fiber space of a linear fiber space, we can consider for each point of the projectivized space $\mathbb{P}(C)$ the corresponding line through the origin in the appropriate fiber of C. We make this precise as follows.

Proposition/Definition 5. *Let X be a scheme and $C \xrightarrow{\pi} X$ a conical fiber space affine over X, with $\mathcal{A}$ the corresponding algebra sheaf on X and $Z = V(\pi^*\mathcal{A}_+)$ the image of the zero section in C, and suppose that $\mathcal{A}$ is generated in degree 1. Then the closed subscheme of $C \times_X \mathbb{P}(C)$ given by the closure of the graph of the quotient map $q : C \setminus Z \to \mathbb{P}(C)$ is a line bundle over $\mathbb{P}(C)$ under the restriction of the natural projection $C \times_X \mathbb{P}(C) \to \mathbb{P}(C)$, called the* **tautological bundle** *of C and denoted $\mathbb{T}(C)$. Indeed, $\mathbb{T}(C)$ is the linear fiber space over $\mathbb{P}(C)$ given by the spectrum $\operatorname{Spec}_+ \mathcal{P}(\mathcal{A}_+(1))$ of the sheaf on $\mathbb{P}(C)$ induced by the irrelevant ideal sheaf of $\mathcal{A}$, considered with the grading which makes it a graded module generated in degree zero.*

(The notation $\mathbb{T}(C)$ is nonstandard, since most authors prefer to talk about locally free sheaves of rank 1 rather than line bundles in our sense; we will encounter the traditional notations below, in Definition 7.)

Proof sketch. All of our claims can be decided affine-locally on X, so we reduce to the case where $X = \operatorname{Spec} R$, $\mathcal{A} = \tilde{S}$ for S an $\mathbb{N}$-graded R-algebra generated in degree 1 with $S_0 = R$, and $C = \operatorname{Spec} S$.

Since S is generated in degree 1, $C \setminus Z$ can be covered by affine open patches $\operatorname{Spec} S_h$ for $h \in S_1$, and as a consequence the corresponding affine open patches $\operatorname{Spec}(S_h)_0$ for such h cover

$\mathbb{P}(C) = \operatorname{Proj} S$. Fix such an h. Then the graph of the quotient map q over $\operatorname{Spec} S_h \subseteq C \setminus Z$ can be viewed as a closed subscheme of the corresponding affine open patch $\operatorname{Spec}(S_h \otimes_R (S_h)_0)$ of $C \times_X \mathbb{P}(C)$, cut out by the ideal $(\frac{g}{h^d} \otimes 1 - 1 \otimes \frac{g}{h^d} \mid d \in \mathbb{N}, g \in S_d)$. The part of the graph's closure in $C \times_X \mathbb{P}(C)$ lying over $\operatorname{Spec}(S_h)_0 \subseteq \mathbb{P}(C)$ is then precisely the closed subscheme of $\operatorname{Spec}(S \otimes_R (S_h)_0)$ cut out by the kernel of the natural map

$$S \otimes_R (S_h)_0 \to \frac{S_h \otimes_R (S_h)_0}{(\frac{g}{h^d} \otimes 1 - 1 \otimes \frac{g}{h^d} \mid d \in \mathbb{N}, g \in S_d)},$$

which is to say the ideal $(g \otimes 1 - h^d \otimes \frac{g}{h^d} \mid d \in \mathbb{N}, g \in S_d)$ — to verify this, we can observe that the codomain is isomorphic to S_h, so that our map can be identified with the multiplication map $S \otimes_R (S_h)_0 \to S_h$, and then work separately in each graded piece to reduce any element of the kernel to zero modulo the stated ideal.

Thus the part of $\mathbb{T}(C)$ lying over $\operatorname{Spec}(S_h)_0 \subseteq \mathbb{P}(C)$ is given by $\operatorname{Spec} \frac{S \otimes_R (S_h)_0}{(g \otimes 1 - h^d \otimes \frac{g}{h^d} \mid d \in \mathbb{N}, g \in S_d)} \cong \operatorname{Spec}(S_h)_0[h]$, and we can see that the transition maps respect the grading and hence are moreover linear fiber space isomorphisms, so that $\mathbb{T}(C)$ is indeed a line bundle over $\mathbb{P}(C)$. To see that it is $\operatorname{Spec}_+ \mathcal{P}(\widetilde{S_+}(1))$ specifically, we note first that $\widetilde{S/S_+}|_{\operatorname{Spec} S_h} \cong 0$ and hence the natural inclusion $\widetilde{S_+} \hookrightarrow \widetilde{S}$ restricts to an isomorphism over $\operatorname{Spec} S_h$. Indeed, since $h \in S_+$, we can see that $(S_+)S_h$ is the unit ideal and we can identify it with $(h) \subseteq S_h$. Thus the degree-zero part of $(S_+)S_h(1)$ is a free module of rank 1 generated by h, and so we have isomorphisms $\operatorname{Spec}_+ \mathcal{P}(\widetilde{S_+}(1))|_{\operatorname{Spec}(S_h)_0} \cong \operatorname{Spec}(S_h)_0[h] \cong \mathbb{T}(C)|_{\operatorname{Spec}(S_h)_0}$. The result now follows by observing that the composed identification is compatible with the transition maps for each line bundle; explicitly, over $\operatorname{Spec}((S_h)_0)_{\frac{h'}{h}} \cong ((S_{h'})_0)_{\frac{h}{h'}}$ for another $h' \in S_1$, we can see in both cases that the transition map $((S_h)_0)_{\frac{h'}{h}}[h] \xrightarrow{\sim} ((S_{h'})_0)_{\frac{h}{h'}}[h']$ is given by extending the usual gluing isomorphism with the assignment $h \mapsto \frac{h}{h'} h'$. □

Example 10. *Let $X = \operatorname{Spec} \mathbb{C}$ and consider $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ as a conical fiber space over X with the usual scaling action, so that $\mathbb{P}(\mathbb{A}^2_{\mathbb{C}}) = \mathbb{P}^1_{\mathbb{C}}$ is the space of lines through the origin in the affine plane. Then $\mathbb{T}(\mathbb{A}^2_{\mathbb{C}})$ is the closed subscheme of $\mathbb{A}^2_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ such that the fiber over each point of $\mathbb{P}^1_{\mathbb{C}}$ is the corresponding line. Explicitly, over the affine patch $\operatorname{Spec} \mathbb{C}[\frac{y}{x}]$ giving the space of non-vertical lines through the origin in the plane, we see from the preceding proof that $\mathbb{T}(\mathbb{A}^2_{\mathbb{C}})$ is given by $\operatorname{Spec} \mathbb{C}[x, y, \frac{y}{x}]/(y - \frac{y}{x}x)$; likewise, above the affine patch $\operatorname{Spec} \mathbb{C}[\frac{x}{y}]$ of non-horizontal lines, we get $\operatorname{Spec} \mathbb{C}[x, y, \frac{x}{y}]/(x - \frac{x}{y}y)$.*

Exercise 4. *Write down explicit chart-wise descriptions of the tautological bundle of $\mathbb{A}^3_{\mathbb{C}}$ as a closed subscheme of $\mathbb{A}^3_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^2_{\mathbb{C}}$.*

Having introduced the tautological bundle arising from a given (suitably well-behaved) conical fiber space, we now note that our discussion in Section 3 implies that each such conical fiber space in fact gives rise to countably many line bundles on its projectivization, the tautological bundles of its Veronese twists. The corresponding sheaves of linear forms (and, dually, of sections) play an important enough role in projective geometry to be named:

Definition 7. *Let X be a scheme and $C \xrightarrow{\pi} X$ a conical fiber space affine over X with the corresponding algebra sheaf $\pi_* \mathcal{O}_C$ generated in degree 1. Then the sheaf $\mathcal{P}((\pi_* \mathcal{O}_C)_+(1))$ of* *linear forms of the tautological bundle $\mathbb{T}(C) \to \mathbb{P}(C)$ is denoted by $\mathcal{O}(C,1)$ and called the* **(Serre) twisting sheaf** *of C.*

More generally, for $n \in \mathbb{Z}$, we define locally free sheaves $\mathcal{O}(C,n)$ of rank 1 on $\mathbb{P}(C)$ as follows:

- $\mathcal{O}(C,0) := \mathcal{O}_{\mathbb{P}(C)}$.
- *For $n > 0$, $\mathcal{O}(C,n) := \mathcal{O}(C\{n\},1)$ is the sheaf of linear forms on the tautological bundle $\mathbb{T}(C\{n\})$ of the nth Veronese twist of C.*
- *For $n < 0$, $\mathcal{O}(C,n) := \mathcal{O}(C,-n)^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}(C)}}(\mathcal{O}(C,-n), \mathcal{O}_{\mathbb{P}(C)})$ is defined to be the dual of $\mathcal{O}(C,-n)$ — that is, the sheaf of sections of $\mathbb{T}(C\{n\})$.*

We call each $\mathcal{O}(C,n)$ the n**th (Serre) twisting sheaf** *of C. In situations where C is understood from context, it is common to drop it from the terminology and notation; that is, we write $\mathcal{O}(n)$ in place of $\mathcal{O}(C,n)$.*

If $\mathcal{F}$ is a quasicoherent sheaf of $\mathcal{O}_{\mathbb{P}(C)}$-modules and $n \in \mathbb{Z}$ is an integer, we define the n**th (Serre) twist** *of $\mathcal{F}$ with respect to C by $\mathcal{F}(C,n) := \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}(C)}} \mathcal{O}(C,n)$; as before, if C is understood from context, we may denote this also by $\mathcal{F}(n)$.*

The algebraic significance of the Serre twisting sheaves and the connection between the apparently different "Serre twists" of Definitions 6 and 7 are explained by the following result:

Proposition 3. *Let X be a scheme and $C \xrightarrow{\pi} X$ a conical fiber space affine over X such that the corresponding algebra sheaf $\mathcal{A} := \pi_*\mathcal{O}_C$ is generated in degree 1. Let $\mathcal{F}$ be a $\mathbb{Z}$-graded quasicoherent sheaf of $\mathcal{A}$-modules. Then, for each $n \in \mathbb{Z}$, there is a natural isomorphism $\mathcal{P}(\mathcal{F})(C,n) \cong \mathcal{P}(\mathcal{F}(n))$ between the nth Serre twist with respect to C of the sheaf on $\mathbb{P}(C)$ induced by $\mathcal{F}$ and the induced sheaf of the nth Serre twist of $\mathcal{F}$ itself.*

In particular, since $\mathcal{O}_{\mathbb{P}(C)} \cong \mathcal{P}(\mathcal{A})$, for each $n \in \mathbb{Z}$ we have $\mathcal{O}(C,n) \cong \mathcal{P}(\mathcal{A}(n))$. As a consequence, we can see for $n, m \in \mathbb{Z}$ that $\mathcal{O}(C,n)(C,m) \cong \mathcal{O}(C,n+m)$.

That is, the algebraic operation of shifting the grading is encoded by the tautological bundles of the quotients $C\{n\}$ of C by the groups of nth roots of unity for integers $n > 0$. As we might expect, there is also a way to see this information more directly from the tautological bundle itself:

Remark 7. *Let X be a scheme and $C \xrightarrow{\pi} X$ a conical fiber space affine over X such that the corresponding algebra sheaf $\pi_*\mathcal{O}_C$ is generated in degree 1. Let $Z_C \hookrightarrow C$ and $Z_{\mathbb{T}(C)} \hookrightarrow \mathbb{T}(C)$ be the zero sections of C and $\mathbb{T}(C)$ over their respective bases, so that $Z_C \cong X$ and $Z_{\mathbb{T}(C)} \cong \mathbb{P}(C)$. Then, by the construction of the tautological bundle, we can see that $C \setminus Z_C \cong \mathbb{T}(C) \setminus Z_{\mathbb{T}(C)}$ as schemes over $\mathbb{P}(C)$; that is, if $q : C \setminus Z_C \to \mathbb{P}(C)$ is the usual quotient map, we have the following commutative diagram:*

$$\begin{tikzcd}
C \setminus Z_C \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, "q"'] & & \mathbb{T}(C) \setminus Z_{\mathbb{T}(C)} \arrow[dl] \\
& \mathbb{P}(C) &
\end{tikzcd}$$

Hence the quotient map q depends only on $\mathbb{T}(C)$, and so we do not need to remember C itself to make use of it. Now, if $\mathcal{G}$ is a quasicoherent sheaf on $\mathbb{P}(C)$, we can see by noting the direct sum decomposition in affine patches as in Remark 6 and using the prior proposition that the Serre twists of $\mathcal{G}$ can be obtained from q; explicitly, we have

$$q_* q^* \mathcal{G} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{G}(n).$$

In particular, if we let $i : C \setminus Z_C \hookrightarrow C$ be the inclusion as in Remark 6 and $\pi' : \mathbb{P}(C) \to X$ be the structure map, we can see by virtue of the fact that $\pi \circ i = \pi' \circ q$ is the natural projection to X that the sheaf $\pi_* i_* q^* \mathcal{G}$ considered in Remark 6 can also be described by

$$\pi_* i_* q^* \mathcal{G} \cong \pi'_* q_* q^* \mathcal{G} \cong \bigoplus_{n \in \mathbb{Z}} \pi'_* \mathcal{G}(n).$$

(As noted in Remark 6, this is most relevant in the case where $\pi_* \mathcal{O}_C$ is moreover locally finitely generated, so that π' is a projective map and each of the sheaves $\pi'_* \mathcal{G}(n)$ is quasicoherent.)

The projection $\mathbb{T}(C) \to C$ arising from our construction of $\mathbb{T}(C)$ as a closed subscheme of $C \times_C \mathbb{P}(C)$ deserves some special attention; as mentioned in the preceding remark, it is as isomorphism over the complement of the zero section Z_C. Over $Z_C \cong X$, on the other hand, we can see that it is given by the natural projection $Z_{\mathbb{T}(C)} \cong \mathbb{P}(C) \to X$. That is, we can think of $\mathbb{T}(C)$ as "a copy of C where the zero section has been replaced by the space of radial directions pointing out from it" (as encoded in our space of "lines through the origin" $\mathbb{P}(C)$); this is our first encounter with the concept of a *blowup*, which we will introduce in full generality next week.

We conclude our discussion of tautological bundles and twisting sheaves by noting that, in the case of projective n-space over a field, *all* line bundles in fact arise from the usual tautological bundle:

Theorem 2. *Let k be a field, and $n > 0$ an integer. Then, if take $\mathbb{P}^n_k$ to be $\mathbb{P}(\mathbb{A}^{n+1}_k)$ and treat the collection of all isomorphism classes of rank-1 locally free sheaves as a group with multiplication $\otimes_{\mathcal{O}_{\mathbb{P}^n_k}}$, the map from $\mathbb{Z}$ to this group given by $n \mapsto \mathcal{O}(n)$ is a group isomorphism.*

This is called the *Picard group* of $\mathbb{P}^n_k$. Picard groups of schemes, especially projective ones, are an important topic in their own right, but we will not discuss them in detail. Note that our result here is particular to fields; if we replace $\operatorname{Spec} k$ by a more general base scheme, it is no longer guaranteed to hold.

5 Projectivizing Line Bundles

As we have seen, projectivization yields "the space of lines through the origin in each fiber of the given conical fiber space". In the case where the conical fiber space in question is a line bundle, so that each fiber is already a single line, projectivization thus yields the base space itself:

Proposition 4. *Let X be a scheme and $L \to X$ a line bundle over X. Then the natural projection is an isomorphism $\mathbb{P}(L) \cong X$ and, moreover, this isomorphism identifies the tautological bundle $\mathbb{T}(L) \to \mathbb{P}(L)$ with L itself, so that the commutative diagram of projections arising from the usual closed embedding $\mathbb{T}(L) \hookrightarrow L \times_X \mathbb{P}(L)$ is as follows:*

$$\begin{tikzcd} \mathbb{T}(L) \arrow[r] \arrow[d, "\wr"'] & \mathbb{P}(L) \arrow[d, "\wr"'] \\ L \arrow[r] & X \end{tikzcd}$$

Proof. The isomorphism $\mathbb{P}(L) \cong X$ can be observed locally; if we take an affine open $\operatorname{Spec} R \subseteq X$ with a trivialization $L|_{\operatorname{Spec} R} \cong \mathbb{A}^1_R = \operatorname{Spec} R[t]$, we see immediately that $\mathbb{P}(L)|_{\operatorname{Spec} R} \cong \operatorname{Proj} R[t] \cong \operatorname{Spec} R$ since $(R[t])_+$ has only one generator, t, and $(R[t]_t)_0 \cong R$. Since the local projections of $\mathbb{P}(L)$ are thus isomorphisms, their transition maps are already determined by the gluings between different affine patches of X, and so we see that the projection $\mathbb{P}(L) \to X$ is an isomorphism globally as well.

Hence we find that the fiber product $L \times_X \mathbb{P}(L)$ is already isomorphic to L under the natural projection; to reach our conclusion, it suffices to note (by again working locally and observing that $R[t] \to R[t]_t$ has trivial kernel, if you like) that the complement of the zero section in L is scheme-theoretically dense, so that the inclusion $\mathbb{T}(L) \hookrightarrow L \times_X \mathbb{P}(L)$ is an equality. $\square$

Thus projectivizations of line bundles are entirely uninteresting in their capacity as schemes over the base space in and of themselves; however, we can exploit the fact that the base space can be reconstructed in this way to apply the machinery we have developed thus far in novel ways. For example, we can generalize our notion of vanishing loci of collections of "functions" on schemes to vanishings of linear forms on arbitrary line bundles:

Definition 8. *Let X be a scheme, $L \to X$ a line bundle, and $\phi : L \to \mathbb{A}^1_X$ a linear form. Then, if $Z \hookrightarrow \mathbb{A}^1_X$ is the zero section, we define the* **vanishing locus** *of ϕ to be the closed subscheme of X given by the projectivization $\mathbb{P}(\phi^* Z)$ of the inverse image $\phi^* Z := L \times_{\mathbb{A}^1_X} Z$ of the zero section of $\mathbb{A}^1_X$ and denote it by $V(\phi)$.*

More generally, if we have a collection $\{\phi_\alpha : L_\alpha \to \mathbb{A}^1_X \mid \alpha \in A\}$ of linear forms on line bundles L_α over X, we define their **(common) vanishing locus** *$V(\phi_\alpha \mid \alpha \in A)$ to be the intersection (in the scheme-theoretic sense of fiber products over X) of the closed subschemes $V(\phi_\alpha)$ of X.*

The notion of the vanishing of a linear form on a line bundle gives us a first step toward the theory of *(Cartier) divisors*, which provides an algebro-geometric analogue to the study of zeroes and poles of rational functions from complex analysis. Our definitions here also admit generalizations in various directions; for example, we can replace $\mathbb{A}^1_X$ by any conical fiber space over X and ϕ by any map of conical fiber spaces (in the sense of preserving the $\mathbb{A}^1_X$-action or in the more permissive sense of Proposition-Definition 3), since all we have really used is the fact that the pullback of the zero section is a sub-conical fiber space of our line bundle. As a case of special interest, we then have a notion of the scheme-theoretic vanishing of a section of a line bundle L, given by the usual identification of sections with linear fiber space maps $\mathbb{A}^1_X \to L$; this agrees with the notion of the vanishing of a section $X \to L$ given by pulling back the zero section along the given one. Indeed, for a given section f of the structure sheaf $\mathcal{O}_X$, the vanishing of f in the usual sense of ideal sheaves will agree with both the vanishing of the corresponding linear form on the trivial line bundle $\mathbb{A}^1_X$ and of the corresponding section of $\mathbb{A}^1_X$, as we should want.

In another direction, we can see that, for each integer $d > 0$, the dth Veronese twist $L\{d\}$ will also be a line bundle over X, and moreover linear forms $L\{d\} \to \mathbb{A}^1_X$ correspond to maps $L \to \mathbb{A}^1_X$ which are conical fiber space maps when twisted by the monoid-with-zero map $\mathbb{A}^1_X \to \mathbb{A}^1_X$ induced by $t \mapsto t^d$ as in Proposition/Definition 3. In particular, we can identify a collection of linear forms on various Veronese twists of L with a collection of homogeneous sections of the corresponding degrees in the symmetric algebra sheaf $\operatorname{Sym} \mathcal{L}(L)$ of the sheaf of linear forms on L; the vanishing of this collection of linear forms is then given by $\operatorname{Proj} \operatorname{Sym} \mathcal{L}(L)/\mathcal{I}$ for $\mathcal{I}$ the homogeneous sheaf of ideals of $\operatorname{Sym} \mathcal{L}(L)$ generated by these sections. Hence, in the case where we consider only linear forms on Veronese twists of a fixed L, our notion of the vanishing of a collection of forms readily generalizes to a notion of the vanishing of *any* homogeneous sheaf of ideals of $\operatorname{Sym} \mathcal{L}(L)$, not just one which is generated by globally-defined sections.

For our purposes, vanishing loci of collections of linear forms on a line bundle will be useful mainly in the context of the following result:

Theorem 3. *Let X be a scheme and $n \geq 0$ an integer. Then there are natural identifications between the following sets:*

- *The set $\operatorname{Hom}_X(X, \mathbb{P}^n_X)$ of maps $X \to \mathbb{P}^n_X$ of schemes over X.*
- *For any scheme S and fixed map $X \to S$, the set $\operatorname{Hom}_S(X, \mathbb{P}^n_S)$ of maps $X \to \mathbb{P}^n_S$ of schemes over S. (Typically, when applicable, we take S to be the spectrum of the ground field.)*
- *The set of choices of a line bundle $L \to X$ and $n+1$ linear forms $\phi_0, \ldots, \phi_n : L \to \mathbb{A}^1_X$ such that $V(\phi_0, \ldots, \phi_n) = \emptyset$, up to isomorphism of line bundles with chosen ordered lists of linear forms.*

Proof sketch. The affine spaces $\mathbb{A}^{n+1}_Y$ (as Y varies over all schemes) arise as pullbacks of $\mathbb{A}^{n+1}_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x_0, \ldots, x_n]$, so that $\mathbb{A}^{n+1}_Y \cong \mathbb{A}^{n+1}_{\mathbb{Z}} \times_{\mathbb{Z}} Y$. In particular, since pullback plays well with composition, we have the usual identification $\mathbb{A}^{n+1}_X \cong \mathbb{A}^{n+1}_S \times_S X$; that is, the trivial rank-$(n+1)$ vector bundle on X is the pullback of the one on S. Since projectivization is a fiberwise operation, this set of facts carries over to projective n-spaces — we have $\mathbb{P}^n_Y \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$ for all schemes Y and hence $\mathbb{P}^n_X \cong \mathbb{P}^n_S \times_S X$. Thus, for any X-scheme X', the set of X-scheme maps $X' \to \mathbb{P}^n_X$ is naturally identified with the set of S-scheme maps $X' \to \mathbb{P}^n_S$ by the universal property of fiber products, and applying this to X in particular gives our identification $\operatorname{Hom}_X(X, \mathbb{P}^n_X) \cong \operatorname{Hom}_S(X, \mathbb{P}^n_S)$. That is, since a map $X \to \mathbb{P}^n_X$ over X — i.e., a section of the natural projection $\mathbb{P}^n_X \to X$ — is already "sending each point of X to a point in the corresponding fiber", both literally and in the stronger scheme-theoretic sense, we lose no information by remembering only the map to, e.g., projective space over a chosen ground field.

It now suffices to construct the identification between $\mathrm{Hom}_X(X, \mathbb{P}^n_X)$ and the set of choices of line bundles with forms described in the theorem statement. On the one hand, we can see from the identification of the tautological bundle $\mathbb{T}(\mathbb{A}^{n+1}_X)$ with the spectrum of the sheaf induced by the irrelevant ideal sheaf in Proposition/Definition 5 that there are linear forms $\xi_0, \ldots, \xi_n : \mathbb{T}(\mathbb{A}^{n+1}_X) \to \mathbb{A}^1_X$ corresponding to its generators $x_0, \ldots, x_n$ and that these satisfy $V(\xi_0, \ldots, \xi_n) = \emptyset$; as such, if $\psi : X \to \mathbb{P}^n_X$ is a map of X-schemes, we have automatically a line bundle $\psi^*\mathbb{T}(\mathbb{A}^{n+1}_X)$ and linear forms $\psi^*\xi_0, \ldots, \psi^*\xi_n$ with $V(\psi^*\xi_0, \ldots, \psi^*\xi_n) = \emptyset$, given by pulling back the whole picture along ψ.

On the other hand, if we are given a line bundle L and linear forms $\phi_0, \ldots, \phi_n : L \to \mathbb{A}^1_X$ with $V(\phi_0, \ldots, \phi_n) = \emptyset$, we can combine the linear forms into a linear fiber space map $\Phi : L \to \mathbb{A}^{n+1}_X$ simply by letting each form give the corresponding coordinate of $\mathbb{A}^{n+1}_X$. The projectivization of this map, as defined in Proposition/Definition 3, is of the form $\mathbb{P}(\Phi) : \mathbb{P}(L) \to \mathbb{P}(\mathbb{A}^{n+1}_X)$ (that is, it is everywhere defined on $\mathbb{P}(L)$) by our condition that the joint vanishing of the chosen forms be empty, and so by Proposition 4 we can regard this as a map $\mathbb{P}(\Phi) : X \to \mathbb{P}^n_X$. That $\mathbb{P}(\Phi)$ depends only on the isomorphism class of our choice of bundle and linear forms can be verified readily from the functoriality of projectivizations of conical fiber space maps — note in particular that an isomorphism of line bundles will induce the identity map on X under projectivization. The argument that this operation and the preceding one are inverse to one another uses the existence of an induced map $\mathbb{T}(\Phi) : \mathbb{T}(L) \to \mathbb{T}(\mathbb{A}^{n+1}_X)$; we omit the details. $\square$

Though it uses a fair amount of machinery, the image behind this fact is rather simple — the forms yield a map of our line bundle into the trivial rank-$(n+1)$ vector bundle over X, which gives us a line through the origin in affine space above each point of X, and this naturally induces a map from X to the space of such lines.

The correspondence between line bundles with chosen sections and maps to projective space is the starting point for the theory of *positivity* of line bundles in algebraic geometry. In particular, if X is a scheme proper over a field k, it is of interest to know whether a given line bundle $L \to X$ admits a choice of $N+1$ linear forms for some N such that the corresponding map $X \to \mathbb{P}^N_k$ is a closed inclusion, and such a bundle is called *very ample*. In practice, one usually works with the theoretically better-behaved class of bundles L such that the mth Veronese twist $L\{m\}$ is very ample for some $m > 0$; these line bundles are said to be *ample*. (As usual, these questions are typically framed in terms of the corresponding sheaf $\mathscr{L}$ on linear forms on L; to define ample bundles, instead of working with the Veronese twists $L\{m\}$, people speak of the tensor powers $\mathscr{L}^{\otimes m}$. Since the objects of study are line bundles, however, it turns out that $\mathscr{L}^{\otimes m}$ is indeed the sheaf of linear forms on $L\{m\}$, although the corresponding statement is certainly not true for quasicoherent sheaves in general or even for higher-rank vector bundles.) These notions form the basis for a rich and interesting theory, which we will not discuss further.

We conclude with a particularly important and well-known instance of a closed embedding given by linear forms on a line bundle:

Proposition/Definition 6. *Let k be a field and fix integers $n \geq 0$ and $d > 0$. Then we can realize $\mathbb{P}^n_k$ as the projectivization of the dth Veronese twist $\mathbb{A}^{n+1}_k\{d\}$ of the usual affine space; under this construction, the tautological bundle $\mathbb{T}(\mathbb{A}^{n+1}_k\{d\})$ comes equipped with $N = \binom{n+d}{n}$ distinguished linear forms, corresponding to the degree-d monomials in the* *variables $x_0, \ldots, x_n$ which generate the irrelevant ideal cutting out the origin in $\mathbb{A}_k^{n+1}\{d\}$. In particular, we can see that the common vanishing locus of these linear forms in $\mathbb{P}_k^n$ is $\emptyset$; moreover, the map $\mathbb{P}_k^n \to \mathbb{P}_k^{N-1}$ they define under the correspondence of Theorem 3 is in fact a closed embedding. This is called the dth* **Veronese embedding** *of $\mathbb{P}_k^n$.*

That is, our quotient $\mathbb{A}_k^{n+1}\{d\}$ of the affine space $\mathbb{A}_k^{n+1}$ by the action of the dth roots of unity can be embedded in a standard way into the higher-dimensional affine space $\mathbb{A}_k^N$ as a closed sub-conical fiber space, and the Veronese embedding arises by projectivizing this inclusion.

Example 11. *In the setting of Example 9, the map of affine schemes corresponding to the ring map $\mathbb{C}[u, v, w] \twoheadrightarrow \mathbb{C}[u, v, w]/(v^2 - uw) \cong \mathbb{C}[x^2, xy, y^2]$ induces the 2nd Veronese embedding $\mathbb{P}_{\mathbb{C}}^1 \hookrightarrow \mathbb{P}_{\mathbb{C}}^2$ when projectivized.*