The Geometry of Rings and Schemes
Lecture 14: Blowups and Normal Cones

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To finish off the semester, we will introduce what is arguably one of the central objects of modern algebraic geometry — a conical fiber space which lurks beneath the surface of many seemingly disparate topics, including birational geometry, intersection theory, local cohomology, and deformation theory. Of course, developing these applications in detail is outside our scope — instead, we will discuss the core constructions and give brief explanations of some of the possible directions one can take them, leaving it to the interested student to explore these topics further through other sources.

1 The Radial Cone

To understand our main definition, we return to the following situation: Let $i : X \hookrightarrow Y$ be a closed inclusion of schemes, so that X is cut out in Y by a quasicoherent sheaf $\mathcal{I}$ of ideals. Then we have the following short exact sequence of quasicoherent sheaves on Y:

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

(Here we really mean $i_*\mathcal{O}_X$ instead of $\mathcal{O}_X$, technically speaking — however, since i is an affine map, we allow ourselves to trim the notation down a little.)

Of course, this gives rise to a corresponding short exact sequence of linear fiber spaces over Y:

$$0 \to \operatorname{Spec}_+ \mathcal{O}_X \to \operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I} \to 0$$

Note that, as usual, we have a change in direction as we pass from algebraic objects to geometric ones. Our goal will be to examine the geometry of this sequence in more detail, using the additional machinery we have developed since we last addressed the topic in Lecture 8.

As we do so, it will useful to have a running example for the sake of illustration:

Example 1. *Let $Y = V(y^2 - x^2(x + 1)) \subseteq \operatorname{Spec} \mathbb{C}[x, y] = \mathbb{A}^2_{\mathbb{C}}$ be a nodal cubic in the affine plane. This curve is regular everywhere except the origin, where it crosses itself, creating a closed singular point. We will take $X = V(x, y) \subseteq Y$ to be this point, endowed with the*

*First draft of the TeX source provided by Márton Beke.

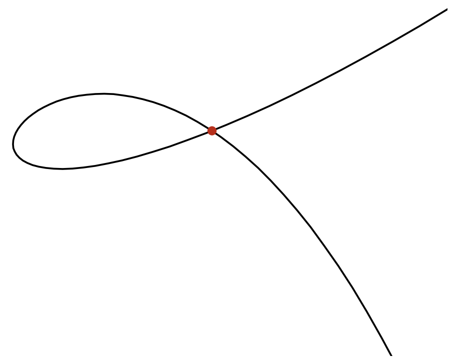

Figure 1: The scheme Y of Example 1, with the subscheme X marked in red. (Y is drawn in perspective, as a curve in the horizontal plane in three-dimensional space, in anticipation of the terrible things we are about to do to it.)

standard reduced scheme structure. This situation is illustrated in Figure 1. In this case, since Y is affine, we can write $\mathcal{I} = \tilde{I}$ for $I = (x, y)$ the maximal ideal at the origin in the ring $R = \mathbb{C}[x, y]/(y^2 - x^2(x + 1))$ of functions on Y.

In our general situation, two of the terms in our short exact sequence are easy to visualize — $\operatorname{Spec}_+ \mathcal{O}_Y$ is simply the trivial line bundle over Y, and $\operatorname{Spec}_+ \mathcal{O}_X$ is a closed sub-linear fiber space which is the trivial line bundle when restricted over X (but not over any larger subscheme with the same underlying set) and zero elsewhere. This is illustrated for the case of Example 1 in Figure 2.

The map $\operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I}$, however, presents greater difficulties. Since pullback of quasicoherent sheaves is not left exact and, correspondingly, pullback of linear fiber spaces is not right exact, the restriction of $\operatorname{Spec}_+ \mathcal{I}$ over a point p need not be the cokernel of the inclusion $\operatorname{Spec}_+ \mathcal{O}_X|_p \hookrightarrow \operatorname{Spec}_+ \mathcal{O}_Y|_p$; that is, our short exact sequence of linear fiber spaces need not be a short exact sequence fiberwise. This can be regarded, in some sense, as a consequence of Nakayama's Lemma — since $\operatorname{Spec}_+ \mathcal{O}_X$ is zero over the open set $Y \setminus X$, while its inclusion into $\operatorname{Spec}_+ \mathcal{O}_Y$ restricts over X to an isomorphism of trivial line bundles, the "fiberwise cokernel" of the inclusion would need to have fiber dimension dropping from 1 over $Y \setminus X$ to 0 over X, a violation of upper semicontinuity.

Now, since restriction of quasicoherent sheaves over open subschemes specifically *is* exact, our short exact sequence and the fact that $\operatorname{Spec}_+ \mathcal{O}_X|_{Y\setminus X}$ is zero give us an isomorphism $\operatorname{Spec}_+ \mathcal{O}_Y|_{Y\setminus X} \cong \operatorname{Spec}_+ \mathcal{I}|_{Y\setminus X}$. That is, $\operatorname{Spec}_+ \mathcal{I}$ is a line bundle — indeed, the trivial one — over the complement of the closed subscheme X defined by $\mathcal{I}$. Its restriction over X, on the other hand, can be given on the level of quasicoherent algebra sheaves on Y as $\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{I} \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathcal{I}) \cong \mathcal{I}/\mathcal{I}^2$ (again, strictly speaking, since we are thinking of this as a sheaf on X, we should write $i^*(\mathcal{I}/\mathcal{I}^2)$). This is to say precisely that $\operatorname{Spec}_+ \mathcal{I}|_X \cong N_{X/Y}$ is the Zariski normal scheme to X in Y, as defined in Lecture 10 — recall that this captures, in some sense, the "perpendicular directions to X in Y".

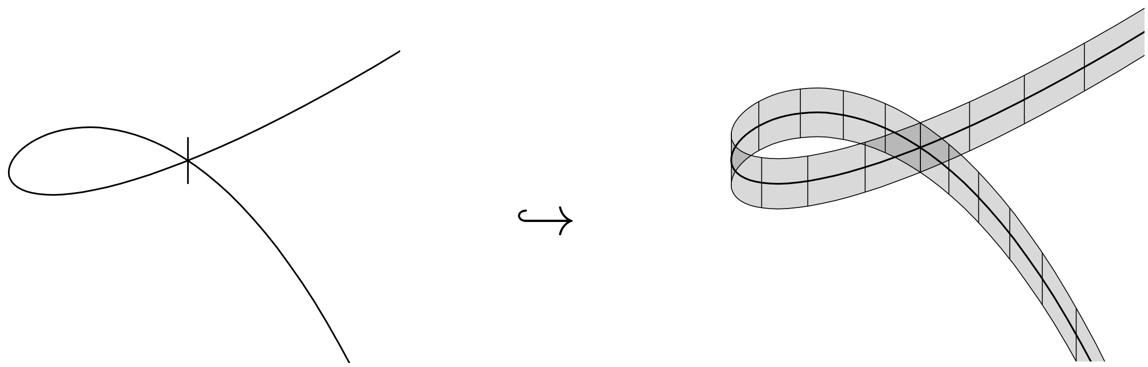

Figure 2: The inclusion of $\operatorname{Spec}_+ \mathcal{O}_X$ (considered over Y — that is, $\operatorname{Spec}_+ i_*\mathcal{O}_X$) into $\operatorname{Spec}_+ \mathcal{O}_Y$ in the situation of Example 1. (Here, and in subsequent figures, we further abuse our beleaguered visual metaphor by truncating the fibers — in this case, we draw closed intervals in place of lines.)

We examine this behavior in the case of our running example:

Example 2. *We continue in the situation of Example 1. Here, since X is a closed point, the Zariski normal scheme $N_{X/Y}$ is in fact the Zariski tangent space to Y at X. In this case, this is the same as the Zariski tangent space $\mathbb{A}^2_{\mathbb{C}}$ to the ambient plane at the origin, since X is a singular point on the plane curve Y. Hence $\operatorname{Spec}_+ \mathcal{I}$ should have a 2-dimensional fiber over the origin and a 1-dimensional fiber over every other point.*

Since Y is an affine scheme, we can consider our map $\operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I}$ on the level of modules — specifically, it is given by the inclusion $(x, y) \hookrightarrow R$. Writing e_1 for x and e_2 for y, we can obtain an explicit presentation $(x, y) \cong \frac{Re_1 \oplus Re_2}{(ye_1 - xe_2, x(x+1)e_1 - ye_2)}$, with the inclusion map now given by mapping each generator to the corresponding variable in R — that is, $e_1 \mapsto x$ and $e_2 \mapsto y$. Taking the symmetric algebras of our modules gives us $\frac{R[e_1,e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2)} \to R[e]$ with $e_1 \mapsto xe$ and $e_2 \mapsto ye$. The corresponding spectra are illustrated in Figure 3.

To understand the map between these spaces, we observe that the intersection of Y with $V(x)$ is set-theoretically the origin — in particular, localizing at x gives the complement $Y \setminus X$. Over this open locus, we can see that $\frac{R_x[e_1,e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2)} \cong R_x[e_1]$ under the map $e_1 \mapsto e_1$, $e_2 \mapsto e_2$; this realizes $\operatorname{Spec}_+ \mathcal{I}|_{Y \setminus X}$ as a trivial line bundle, as claimed. The map from $\operatorname{Spec}_+ \mathcal{O}_Y|_{Y \setminus X} \cong \operatorname{Spec} R_x[e]$ then corresponds to the R_x-algebra map $R_x[e_1] \to R_x[e]$ given by $e_1 \mapsto xe$.

Hence the map $\operatorname{Spec}_+ \mathcal{O}_Y|_{Y \setminus X} \to \operatorname{Spec}_+ \mathcal{I}|_{Y \setminus X}$ is given by scaling each fiber by x; intuitively, as we approach the origin, we are contracting the fibers of the trivial line bundle by greater and greater amounts. In the "limit", our map contracts the fiber of the trivial bundle over X to zero (since the images xe and ye of e_1 and e_2 are both zero over X, by design), as one would expect — indeed, by the universal property of the cokernel, this is the unique map out of the trivial line bundle such that every other such map of quasicoherent linear fiber spaces over Y contracting the fiber over X to zero factors uniquely through it.

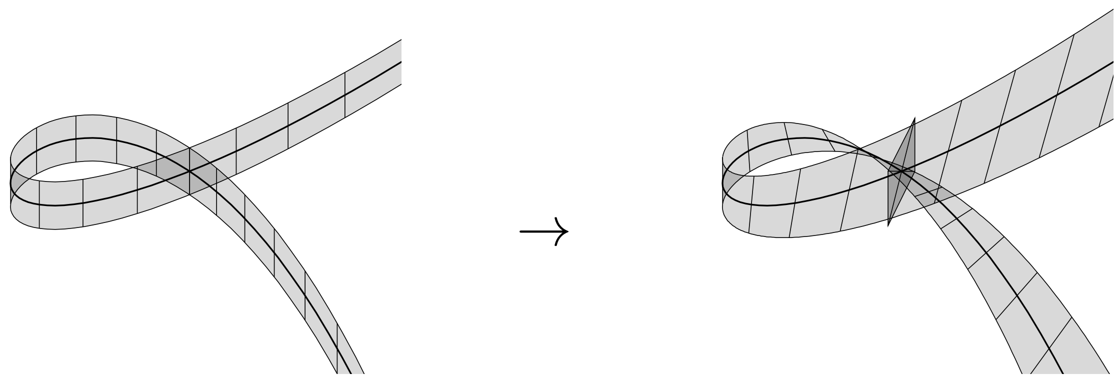

Figure 3: The spaces $\operatorname{Spec}_+ \mathcal{O}_Y = \operatorname{Spec} R[e]$ and $\operatorname{Spec}_+ \mathcal{I} = \frac{R[e_1,e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2)}$ in the situation of Examples 1 and 2. (In the latter case, the necessity of embedding the picture in three-dimensional space introduces some fictitious self-intersections — hopefully, the correct interpretation is clear from the fact that $\operatorname{Spec}_+ \mathcal{I}$ is a linear fiber space over Y.)

For the sake of concreteness, we now make some observations in the setting of our ongoing example. As already mentioned, the fiber of $\operatorname{Spec}_+ \mathcal{I}$ over our point X is the Zariski tangent space to Y at X; however, our picture in Figure 3 does not make this relationship particularly clear, and so we illustrate the Zariski tangent space of the node at the origin separately in Figure 4.

From this image, we can see that, although all of the tangent directions to the affine plane at the origin X are also "tangent" to Y in the sense of Zariski tangent spaces (as they indeed must be, since X is a singular point of Y and $\dim Y = \dim \mathbb{A}^2_{\mathbb{C}} - 1$), two directions are very clearly "more tangent" than the others — the "limiting tangent lines" along the two "local branches" of the curve. (Here we must make generous use of quotation marks, at least for the time being — as discussed in Lecture 11, the classical notions of limits and convergence don't really capture the correct geometric intuitions in the scheme setting if we apply them naively, and the talk of local branches doesn't make sense because our curve consists of a single irreducible component, even if we pass to the germ at the origin. This latter issue can be surmounted through the machinery of *completion*, which we will not discuss this semester.) The remaining directions are, in some sense, artificial — they arise essentially because the Zariski tangent space is a vector space (i.e., a linear fiber space over the corresponding closed point) and hence must contain the linear span of the two "actual tangent directions".

Relatedly, we can see in Figure 3 that, in the case of our running example, the fiber over X is its own irreducible component — one which seems to be, again, more an artifact of the fact that $\operatorname{Spec}_+ \mathcal{I}$ is a linear fiber space than something which really properly arises from the relative geometry of X and Y. Here we can be more precise about the nature of this incongruity by examining the map $\operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I}$. Now, this map of linear fiber spaces is the right-hand nonzero map in a short exact sequence, which is to say that it plays the same role as a quotient map in the setting of modules. From the module setting, our expectation is that such maps should be surjective — however, as we have already seen in

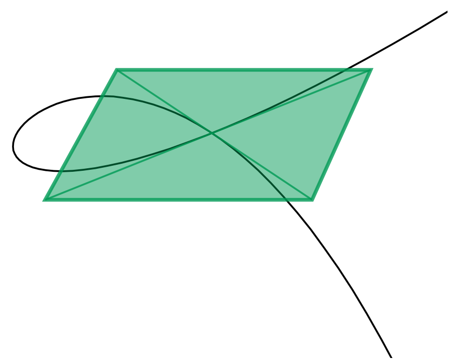

Figure 4: The scheme Y of Examples 1 and 2, with the Zariski tangent space at the closed point X drawn in the plane in green. Here we also indicate the "limiting" tangent directions with green lines.

Example 2, this one collapses the fiber of $\operatorname{Spec}_+ \mathcal{O}_Y$ over the origin to the corresponding point of the zero section of $\operatorname{Spec}_+ \mathcal{I}$, and in particular the image of the induced map on the underlying topological spaces does not contain any of the other points of $\operatorname{Spec}_+ \mathcal{I}$ lying in the fiber over the origin.

As we have remarked in the past, as far back as Lecture 2, there is no good notion of an "image" of a map of rings or schemes as a scheme. However, in the case of an affine map, there *is* a good notion of the closure of its image — that is, the smallest closed subscheme of the target through which the map factors. Concretely, if the affine map is given as the (structure map of the) relative spectrum of the quasicoherent $\mathcal{O}_S$-algebra sheaf $\mathcal{A}$ (for S the target scheme), then the closure of the image is precisely the subscheme cut out by the kernel of the sheaf map $\mathcal{O}_S \to \mathcal{A}$ giving the algebra structure on $\mathcal{A}$. Maps between linear fiber spaces being affine, this means that we have a well-defined notion of the closure of the image of $\operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I}$, and we can see that even this is not all of $\operatorname{Spec}_+ \mathcal{I}$, despite the "surjectivity" we might expect from the exactness of $\operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I} \to 0$; because $\operatorname{Spec}_+ \mathcal{I}$ satisfies the categorical properties we should expect from a quotient only in the context of linear fiber spaces over Y, not Y-schemes more generally, nothing about this exactness precludes the existence of a proper closed subscheme through which our map factors, provided it is not itself a linear fiber space.

Hence we have a precise way of distinguishing from the "more natural" parts of $\operatorname{Spec}_+ \mathcal{I}$ and those which we think of as "artifacts of linearity"; the former will comprise the closure of the image of the map $\operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I}$ and the latter the complement of this closed subscheme in the whole linear fiber space. Indeed, this turns out to capture, as well, our notion of the "more tangent" directions depicted in Figure 4, both in the specific case of our running example and in general. We therefore give this closed subscheme of $\operatorname{Spec}_+ \mathcal{I}$ a name:

Definition 1. *As above, let $i : X \hookrightarrow Y$ be a closed inclusion of schemes, with $\mathcal{I}$ the* *corresponding ideal sheaf on Y. Then the quasicoherent $\mathcal{O}_Y$-algebra sheaf given by*

$$\mathcal{B}_X Y := \frac{\operatorname{Sym}\mathcal{I}}{\ker(\operatorname{Sym}\mathcal{I} \to \operatorname{Sym}\mathcal{O}_Y)} = \operatorname{im}(\operatorname{Sym}\mathcal{I} \to \operatorname{Sym}\mathcal{O}_Y)$$

(where the map $\operatorname{Sym}\mathcal{I} \to \operatorname{Sym}\mathcal{O}_Y$ is the one arising from the $\mathcal{O}_Y$-module sheaf inclusion $\mathcal{I} \to \mathcal{O}_Y$; explicitly, under the identification $\operatorname{Sym}\mathcal{O}_Y \cong \mathcal{O}_Y[t]$, we then have $\mathcal{B}_X Y = \bigoplus_{k=0}^{\infty} \mathcal{I}^k t^k \subseteq \bigoplus_{k=0}^{\infty} \mathcal{O}_Y t^k = \mathcal{O}_Y[t]$) is called the **blowup algebra sheaf** *of X in Y. We will call its relative spectrum $R_X Y := \operatorname{Spec}\mathcal{B}_X Y$ the* **radial cone** *of X in Y.*

(The term "blowup algebra sheaf" is reasonably standard, while "radial cone" is not. We will discuss the motivations for both of them shortly.)

By construction, the radial cone comes equipped with natural maps $\operatorname{Spec}_+ \mathcal{O}_Y \to R_X Y \hookrightarrow \operatorname{Spec}_+ \mathcal{I}$ factoring $\operatorname{Spec}_+ \mathcal{O}_Y \to \operatorname{Spec}_+ \mathcal{I}$, with the latter a closed inclusion, the minimal one for which such a factorization exists. We can also see that, since the map $\operatorname{Sym}\mathcal{I} \to \operatorname{Sym}\mathcal{O}_Y$ preserves the grading, its kernel is a homogeneous sheaf of ideals of $\operatorname{Sym}\mathcal{I}$, and so, as the name would suggest, the radial cone $R_X Y$ is in fact a closed sub-conical fiber space of $\operatorname{Spec}_+ \mathcal{I} := \operatorname{Spec}\operatorname{Sym}\mathcal{I}$, not just a closed subscheme.

Remark 1. *In fact, the construction makes it clear that this kernel is generated in degrees strictly larger than 1, since the map $\operatorname{Sym}\mathcal{I} \to \operatorname{Sym}\mathcal{O}_Y$ is given in degree 1 by the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. Hence the smallest closed sub-linear fiber space of $\operatorname{Spec}_+ \mathcal{I}$ containing $R_X Y$ is $\operatorname{Spec}_+ \mathcal{I}$ itself.*

We compute the radial cone in the case of our running example:

Example 3. *We continue in the situation of Examples 1 and 2. As noted in Example 2, the map $\operatorname{Sym}\mathcal{I} \to \operatorname{Sym}\mathcal{O}_Y$ in this case is given by (the map of quasicoherent sheaves on the affine scheme Y corresponding to) the R-algebra morphism $\frac{R[e_1,e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2)} \to R[e]$ taking e_1 to xe and e_2 to ye; its kernel turns out to be the ideal $((x+1){e_1}^2 - {e_2}^2)$. Hence we have the factorization*

$$\operatorname{Sym} I \twoheadrightarrow \frac{R[e_1, e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2, (x+1){e_1}^2 - {e_2}^2)} \hookrightarrow \operatorname{Sym} R,$$

which gives the factorization $\operatorname{Spec}_+ \mathcal{O}_Y \to R_X Y \hookrightarrow \operatorname{Spec}_+ \mathcal{I}$ after taking spectra. (Note that here the ring R and the "R" in the notation $R_X Y$ are unrelated!) The inclusion of the radial cone $R_X Y$ into $\operatorname{Spec}_+ \mathcal{I}$ is illustrated in Figure 5.

In particular, the restriction $R_X Y|_X$ of the radial cone will give a closed subscheme of the Zariski tangent space $N_{X/Y} \cong \operatorname{Spec}\left(\frac{R[e_1,e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2)} \otimes_R R/(x,y)\right) \cong \operatorname{Spec}\mathbb{C}[e_1, e_2]$, cut out by the ideal $((x+1){e_1}^2 - {e_2}^2) = ({e_1}^2 - {e_2}^2) = ((e_1 - e_2)(e_1 + e_2))$ of $\mathbb{C}[e_1, e_2]$. This is exactly the union of the two "limiting tangent lines", as depicted in Figure 4.

More generally, we can see that the restriction of the radial cone $R_X Y$ of X in Y over X will be a closed subscheme of the Zariski normal scheme $N_{X/Y}$. As in the case of our running example, we can think of this as giving the "normal directions" to our subscheme X which "really arise from the ambient scheme Y", rather than being mere artifacts of linearity — for a perspective on how to make this characterization mathematically precise, see Remark 2. This restriction of the radial cone over the closed subscheme in question is important enough to have its own name:

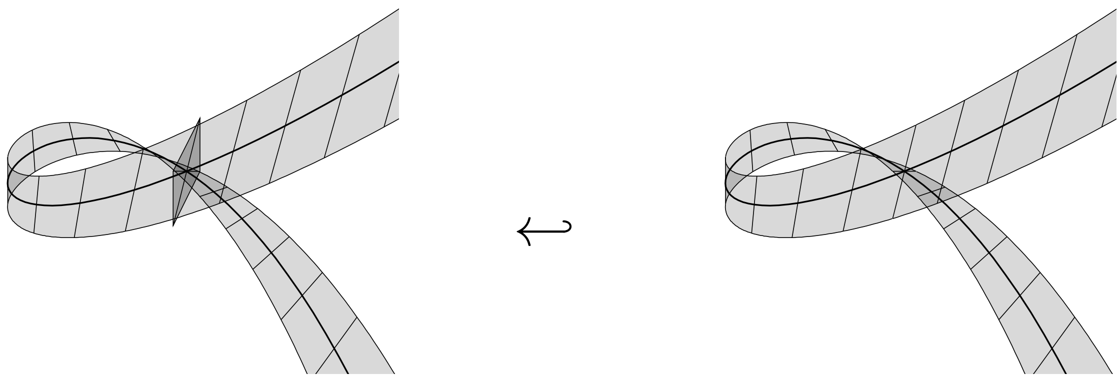

Figure 5: The inclusion of the radial cone $R_X Y$ into $\operatorname{Spec}_+ \mathcal{I}$ in the situation of Examples 1, 2, and 3, where it corresponds to the quotient map $\frac{R[e_1,e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2)} \twoheadrightarrow \frac{R[e_1,e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2, (x+1)e_1{}^2 - e_2{}^2)}$. (As before, our manner of depicting our schemes in three-dimensional space suggests more self-intersections than are actually present; in this case, if we think of the visual representation of $R_X Y$ as a strip of ribbon, it should meet itself at only one point, the node of the curve in the zero section.)

Definition 2. *As above, let $i : X \hookrightarrow Y$ be a closed inclusion of schemes, with $\mathcal{I}$ the corresponding ideal sheaf on Y. Then we call $C_X Y := R_X Y|_X$ the* **normal cone** *of X in Y; concretely, this is the relative spectrum of the* **associated graded algebra sheaf**

$$\operatorname{gr}_{\mathcal{I}} \mathcal{O}_Y := \bigoplus_{k=0}^{\infty} (\mathcal{I}^k/\mathcal{I}^{k+1}) t^k = \mathcal{B}_X Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X.$$

(To be precise, since we view this as a conical fiber space over X, we should really refer to $i^*(\mathcal{I}^k/\mathcal{I}^{k+1})$ and $i^*(\mathcal{B}_X Y \otimes_{\mathcal{O}_Y} i_* \mathcal{O}_X)$, or simply $i^*\mathcal{I}^k$ and $i^*\mathcal{B}_X Y$, but we simplify our notations as usual.)

In the case where X is a single point with the reduced scheme structure, we also call $C_X Y$ the **tangent cone** *of Y at X.*

As we have already mentioned, the normal cone carries a natural embedding $C_X Y \hookrightarrow N_{X/Y}$ into the Zariski normal scheme as a closed subscheme — indeed, as a closed sub-conical fiber space, since the radial cone is a sub-conical fiber space of $\operatorname{Spec}_+ \mathcal{I}$.

Unlike "radial cone", the term "normal cone" is entirely standard, and the embedding of the normal cone into the Zariski normal scheme gives us a clear picture of the normal cone in terms of the geometry of the inclusion $X \hookrightarrow Y$, as discussed. To extend this picture to the whole radial cone, recall that $R_X Y$ is a line bundle over $Y \setminus X$; that is, over any point x of X the fiber (of either cone) is a union of lines through the origin in $N_{X/Y}|_x$ giving the "limiting normal directions to X in Y at x", whereas over a point $y \in Y \setminus X$ the fiber of the radial cone is a single line, which we think of as the direction pointing "radially inward to/outward from X", so that it makes sense for "limits" of these lines as we approach X to give "directions normal to X". This is the motivation for the term "radial cone". Of course, as in the original case of Zariski normal schemes, our "directions" here are directions only in a fairly abstract sense — nothing in our discussion gives an embedding into the tangent scheme of Y over any particular base, for example.

Since we are speaking of the fibers of the radial cones as the lines corresponding to certain directions, it also makes sense to talk about the corresponding spaces of directions themselves — that is, about the fibers of the projectivization of the radial cone. As it turns out, this projectivization, like the restriction of the radial cone over X, actually has a widely-used name:

Definition 3. *Let $X \hookrightarrow Y$ be a closed inclusion of schemes. Then we call the projectivization*

$$\operatorname{Bl}_X Y := \mathbb{P}(R_X Y) = \operatorname{Proj} \mathcal{B}_X Y$$

of the radial cone of X in Y the **blowup** *of X in Y (or of Y at X).*

Since the radial cone is a line bundle over $Y \setminus X$ and a line bundle, as discussed last week, gives a copy of the base scheme when projectivized, the structure map $\operatorname{Bl}_X Y \to Y$ is an isomorphism over $Y \setminus X$. On the other hand, we can see that the restriction over X is exactly the projection $\mathbb{P}(C_X Y) \to Y$ of the projectivized normal cone (which we typically call the *exceptional divisor* when we regard it as a subscheme of the blowup), giving the space of limiting normal directions to X over each point. That is, we can think of the blowup operation as follows: We start with Y, remove X from it, and install the projectivized normal cone $\mathbb{P}(C_X Y)$ in its place, gluing $Y \setminus X$ to this space according to the limiting radial directions.

We illustrate this in the case of our running example:

Example 4. *We continue in the situation of Examples 1, 2, and 3. By our computation of the radial cone in Example 3, the blowup of our nodal cubic Y at the point X will be given by*

$$\operatorname{Bl}_X Y = \operatorname{Proj} \frac{R[e_1, e_2]}{(ye_1 - xe_2, x(x+1)e_1 - ye_2, (x+1){e_1}^2 - {e_2}^2)}.$$

Since the zero section of the radial cone in this case is cut out by the ideal (e_1, e_2), we can compute this projectivization by considering only the two affine patches given by the non-vanishing loci of e_1 and e_2. However, since $(x+1){e_1}^2 = {e_2}^2$, we can see that inverting e_2 will also create an inverse, $(x+1)e_1{e_2}^{-2}$, for e_1 — that is, the localization at e_2 factors through the localization at e_1, and so the non-vanishing locus of e_2 is contained in that of e_1. As such, the whole blowup will itself be affine, given by

$$\operatorname{Bl}_X Y = \operatorname{Spec} \left(\frac{R[e_1, e_2]_{e_1}}{(ye_1 - xe_2, x(x+1)e_1 - ye_2, (x+1){e_1}^2 - {e_2}^2)} \right)_0;$$

note in particular that the structure map of the blowup in this case is both projective and affine, hence finite. Writing $\lambda := \frac{e_2}{e_1}$, we then have

$$\operatorname{Bl}_X Y = \operatorname{Spec} \frac{R[\lambda]}{(y - x\lambda, x(x+1) - y\lambda, (x+1) - \lambda^2)} = \operatorname{Spec} \frac{\mathbb{C}[x, y, \lambda]}{(x - (\lambda^2 - 1), y - \lambda(\lambda^2 - 1))},$$

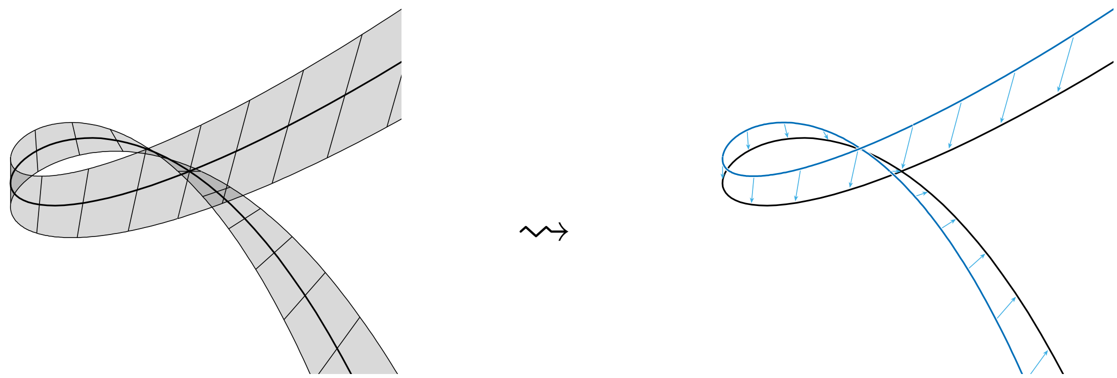

Figure 6: The radial cone of X in Y and its projectivization, the blowup $\mathrm{Bl}_X Y$ of X in Y, in the situation of Examples 1, 2, 3, and 4. In the right-hand picture, we illustrate the blown-up curve in blue and draw arrows illustrating the projection to the original nodal cubic in light blue. (Note that the blowup is nonsingular in this case, even though it passes over itself when drawn in perspective.)

which is isomorphic to the affine line $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[\lambda]$. Hence we can see that the structure map of the blowup is given by the $\mathbb{C}$-algebra map $\frac{\mathbb{C}[x,y]}{(y^2-x^2(x+1))} \to \mathbb{C}[\lambda]$ taking x to $\lambda^2 - 1$ and y to $\lambda(\lambda^2-1)$ in this case.

The geometric picture is illustrated in Figure 6. Here the blowup turns out to provide a parameterization of our original nodal curve Y, which is mostly one-to-one — the fiber over the point X at which we blew up, however, consists of two points, one for each of the limiting directions along which we can approach it in Y. (Equivalently: One for each of the lines through the origin in the tangent cone $C_X Y = \operatorname{Spec} \mathbb{C}[e_1, e_2]/({e_1}^2 - {e_2}^2)$ we computed in Example 3.)

(Note that, although the blowup of Y at X in this case turns out to be nonsingular, blowups of singular schemes can remain singular in general. We will discuss the relationship between blowups and singularities further in Section 2.)

Our discussion thus far has made heavy use of informal notions of “limiting behavior” which, as mentioned, are not borne out in the literal topology of our schemes' underlying spaces. As we saw in Lecture 11, the way to make such “convergence-theoretic” concepts precise in the setting of schemes is usually to phrase them in valuative terms, which is possible here (at least in nice cases):

Remark 2. *As above, let $i : X \hookrightarrow Y$ be a closed inclusion of schemes, with $\mathcal{I}$ the corresponding ideal sheaf on Y. Suppose further that Y is locally Noetherian, so that $\mathbb{P}(\operatorname{Spec}_+ \mathcal{I}) \to Y$ and $\mathrm{Bl}_X Y \to Y$ are projective maps. Let K be a field with a discrete valuation v, so that the spectrum $\operatorname{Spec} \mathcal{O}_v$ of the corresponding discrete valuation ring is a nonsingular curve germ, and suppose we have a map $\mu : \operatorname{Spec} \mathcal{O}_v \to Y$ which does not factor through our inclusion $i : X \hookrightarrow Y$ — that is, μ gives a parameterized curve germ in Y which is not contained in X.*

$$\begin{tikzcd}
 & \mathrm{Bl}_X Y \arrow[r, hook] \arrow[d] & \mathbb{P}(\mathrm{Spec}_+ \mathcal{I}) \arrow[dl] \\
\mathrm{Spec}\,\mathcal{O}_v \arrow[r, "\mu"] & Y &
\end{tikzcd}$$

Now, since μ does not factor through the closed subscheme X, the generic point $\mathrm{Spec}\,K$ of $\mathrm{Spec}\,\mathcal{O}_v$ must be sent to a point of $Y \setminus X$. Since both $\mathrm{Spec}_+ \mathcal{I}$ and $R_X Y$ are line bundles over $Y \setminus X$ (and, indeed, the inclusion $R_X Y \hookrightarrow \mathrm{Spec}_+ \mathcal{I}$ is an isomorphism over this locus), their projectivizations' restrictions over $Y \setminus X$ map to it isomorphically and so we can compose with the inverse of this isomorphism to obtain a map $\mathrm{Spec}\,K \xrightarrow{\omega} \mathrm{Bl}_X Y|_{Y\setminus X} = \mathbb{P}(\mathrm{Spec}_+ \mathcal{I})|_{Y \setminus X}$ lifting $\mu|_{\mathrm{Spec}\,K}$:

$$\begin{tikzcd}
\mathrm{Spec}\,K \arrow[r, "\omega"] \arrow[d, hook] & \mathrm{Bl}_X Y \arrow[r, hook] \arrow[d] & \mathbb{P}(\mathrm{Spec}_+ \mathcal{I}) \arrow[dl] \\
\mathrm{Spec}\,\mathcal{O}_v \arrow[r, "\mu"] \arrow[ur, dashed, "\exists!\bar{\omega}"] & Y &
\end{tikzcd}$$

Since projective maps are proper, we can then see that μ lifts to a unique map $\bar{\omega}$: $\mathrm{Spec}\,\mathcal{O}_v \to \mathrm{Bl}_X Y$ by the valuative criterion; if μ does not factor through the inclusion of $Y \setminus X$ into Y, so that μ defines a parameterized curve germ set-theoretically meeting X only at its closed point, we can see that the image of the closed point under $\bar{\omega}$ will be a point of the projectivized normal cone $\mathbb{P}(C_X Y)$, and we can think of this (or the corresponding line through the origin in $C_X Y$) as the "limiting normal direction along the curve parameterized by μ". This makes sense of our claim that blowing up amounts to replacing X by its projectivized normal cone with a gluing given by the limiting radial directions.

Likewise, our characterization of the normal cone $C_X Y$ as the subspace of $N_{X/Y}$ given by the normal directions which are actually attained can now be understood in these terms. Since the structure map $\mathbb{P}(\mathrm{Spec}_+ \mathcal{I}) \to Y$ is proper, we can see that we could have taken $\bar{\omega}$ as a map to $\mathbb{P}(\mathrm{Spec}_+ \mathcal{I})$, even before the defining the blowup — however, by the uniqueness of lifts guaranteed by the valuative criterion, we can see that this lift of μ is necessarily the composition of our original $\bar{\omega} : \mathrm{Spec}\,\mathcal{O}_v \to \mathrm{Bl}_X Y$ with the inclusion $\mathrm{Bl}_X Y \hookrightarrow \mathbb{P}(\mathrm{Spec}_+ \mathcal{I})$, and so we find that all the points of $\mathbb{P}(N_{X/Y})$ which are actually "limits of radial directions along curves" in this sense must already lie in $\mathbb{P}(C_X Y)$.

Our valuative picture is not perfectly comprehensive, however — in particular, maps from the nonsingular curve $\mathrm{Spec}\,\mathcal{O}_v$ to a scheme S necessarily factor through the reduction S_{red}, so our discussion above is blind to all nonreduced behavior. Hence we cannot say in general that, e.g., the blowup is the smallest closed subscheme of $\mathbb{P}(\mathrm{Spec}_+ \mathcal{I})$ through which all lifted maps from curve germs factor in this way, since this may instead be the smaller scheme $(\mathrm{Bl}_X Y)_{\mathrm{red}}$. Nevertheless, the valuative criterion offers a reasonable first step in making our intuitions about limiting behavior precise.

We conclude our initial discussion by noting that, as mentioned last week, the tautological bundle of conical fiber space can be regarded as the blowup of the conical fiber space itself at the zero section:

Proposition 1. *Let X be a scheme and $C \xrightarrow{\pi} X$ a conical fiber space affine over X, with $\mathcal{A}$ the corresponding algebra sheaf on X and Z the image of the zero section in C, and suppose that $\mathcal{A}$ is generated in degree 1. Then $\mathbb{T}(C) \cong \mathrm{Bl}_Z C$ as schemes over C.*

Proof sketch. As usual, we work affine-locally on X, taking $X = \operatorname{Spec} R$, $\mathcal{A} = \tilde{S}$ for S an $\mathbb{N}$-graded R-algebra generated in degree 1 with $S_0 = R$, and $C = \operatorname{Spec} S$. Then the blowup algebra sheaf of Z in C is the sheafification of the S-algebra $\bigoplus_{k=0}^{\infty} (S_+)^k t^k$.

We construct a map $\mathrm{Bl}_Z C \to \mathbb{T}(C)$ as follows. First observe that there are two natural maps $S \to \bigoplus_{k=0}^{\infty} (S_+)^k t^k$ we can consider; the first, corresponding to the typical structure map of the radial cone, takes S isomorphically to the degree-zero part $(S_+)^0 t^0 \cong S$, while the second is given by $h \mapsto t^d h$ for each $d \in \mathbb{N}$ and $h \in S_d$. Observe that this latter map is degree-preserving; indeed, we can see that it arises by factoring the monoid-with-zero multiplication $\mathbb{A}^1_C \to C$ through the natural map $\mathbb{A}^1_C \cong \operatorname{Spec}_+ S \to R_Z C$ given by the module inclusion $S_+ \hookrightarrow S$. Geometrically, the corresponding map $R_Z C \to C$ is then given away from Z by mapping the fiber of $R_Z C$ over each point to the corresponding line through the origin in C; over Z, it maps each line through the origin in the normal cone $C_Z C$ (Note that here the two "C"s refer to different things!) to the line through the origin in C which gives the corresponding normal direction under the process discussed in Remark 2. That is, this map is exactly the one which sends each "abstract radial line" in $R_Z C$ to the $\mathbb{A}^1_R$-orbit in C which realizes it.

By combining these two maps in the order given, we obtain from the universal property of the fiber product a single map $R_Z C \to C \times_X C$ which is moreover a map of conical fiber spaces in the sense of commuting with the monoid-with-zero actions, where we consider the product as a conical fiber space over C using the projection to the first factor. Indeed, we can see by considering the corresponding algebra quotients that the pullback along this map of the zero section $C \times_X Z$ is exactly the zero section of $R_Z C$, and so we can projectivize to obtain a corresponding map $\mathrm{Bl}_Z C \to C \times_X \mathbb{P}(C)$. The result now follows by observing that the given algebra map is surjective and so the map of schemes is a closed inclusion; we can verify chartwise that the ideal sheaf cutting it out is exactly the one used to define $\mathbb{T}(C)$. $\square$

Our method of proof here provides some further support for our intuition for $R_X Y$ in terms of radial directions — in this case, the notion of a "direction radiating outward from the zero section" can be made very precise using (the appropriate map of tangent schemes induced by) the $\mathbb{A}^1$-action, and, as discussed, our map from the radial cone then takes each abstract radial line to the $\mathbb{A}^1$-orbit realizing it.

2 Some Applications of Normal Cones and Blowups

Explicit discussion of radial cones *per se* is fairly rare — so much so, as mentioned, that there is no standard name for them — but the normal cones and blowups we obtain from them are ubiquitous in algebraic geometry. Here, without necessarily going into too much detail, we survey some significant concepts and subfields which make use of these objects.

2.1 Multiplicity

The first of these requires the notion of the *degree* of a closed embedding into $\mathbb{P}^n_k$ (for k a field and $n \geq 0$ an integer), which we left out of our discussion of projective space last week. We will not go into the machinery required to cover this concept in detail — suffice it to say that it is a numerical quantity giving, in an appropriate sense, the number of points in the vanishing of $\dim X$ generically chosen k-linear combinations of the linear forms defining the embedding map and that this generalizes the degree of the defining equation in the case where our closed subscheme is cut out by a single homogeneous polynomial. For our purposes, the interesting thing about this construction is that it allows us to obtain numerical information about the structure of an arbitrary locally Noetherian scheme at a given point:

Definition 4. *Let X be a locally Noetherian scheme and x a point of X where the local dimension of X is positive. By replacing X with $\operatorname{Spec} \mathcal{O}_{X,x}$ as necessary, we can suppose that x is, in particular, a closed point of X. Then the tangent cone C_xX of X at x is by definition a closed subscheme of the Zariski tangent space $N_{x/X}$ of X at x; by projectivizing, we obtain a closed embedding $\mathbb{P}(C_xX) \hookrightarrow \mathbb{P}(N_{x/X}) \cong \mathbb{P}^n_{\kappa(x)}$ for some integer $n \geq 0$. We then define the* **multiplicity** *of X at x to be the degree of this embedding.*

This provides a very coarse quantification of the extent to which X is singular at x; in nice circumstances, for example, it turns out to be true that x is a singular point of X if and only if the multiplicity of X at x is larger than 1. Multiplicity can thus be a valuable starting point in, e.g., classification results for singularity germs satisfying specified conditions.

2.2 Birational Geometry and Resolution of Singularities

Our second application deals with partially-defined maps between schemes — that is, "maps" which are defined only on, say, a dense open subset of their domains; in some circumstances, it can be more useful to work with these than with maps in the more traditional sense. Last week, for example, we saw that a map $C \to C'$ of conical fiber spaces over a given base scheme may not induce a corresponding map $\mathbb{P}(C) \to \mathbb{P}(C')$ of projectivizations — rather, the induced map is defined only on the complement in $\mathbb{P}(C)$ of the projectivization of the pullback of the zero section of C'. Hence, so long as we consider only conical fiber space maps which do not map any irreducible component of the source into the zero section of the target, we can work in the following setting:

Definition 5. *Let X and Y be schemes. The collection of* **rational maps** *from X to Y is the set of pairs (U, f) such that U is a (set-theoretically) dense open subscheme of X and $f : U \to Y$ is a map of schemes, modulo the equivalence relation which identifies (U, f) and (V, g) if there is some (set-theoretically) dense open W contained in $U \cap V$ such that $f|_W = g|_W$. (If we use scheme-theoretic density instead of set-theoretic, we obtain the notion of a pseudo-morphism from X to Y; in general, rational maps are most often studied in settings where the schemes in question are assumed to be reduced, so that these two definitions coincide.) We denote a rational map by $f : X \dashrightarrow Y$.*

In general, it may not be possible to compose two rational maps and obtain a third rational map, since the pullback of a dense open subscheme along a given map of schemes is *not guaranteed to be dense. There are several ways to address this issue, in varying levels of generality, but for simplicity we proceed as follows: Further require that X and Y be irreducible and call a rational map $f : X \dashrightarrow Y$* **dominant** *if it sends the generic point of X to that of Y — since the generic point is contained in every dense open, this property is independent of the chosen representative. (Equivalently: The set-theoretic image of any/every representative of f is dense in Y.)*

We can then consider the category where the objects are, say, integral schemes and the morphisms are dominant rational maps. Invertible maps in this category are called **birational maps***, and integral schemes which are isomorphic to each other in this category are often said to be* **birationally equivalent***, or simply* **birational***, to one another.*

Such maps are called "rational" by reference to the case of, for example, maps $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$; in this setting, "rational maps" in the sense defined above are given by rational functions in the traditional sense of ratios of complex polynomials. More generally, if R is a domain, rational maps from $\operatorname{Spec} R$ to the affine line can be identified with elements of the field of fractions of R.

The study of, say, integral finite-type separated schemes over a field up to birational equivalence is called *birational geometry*. For example, a very famous and longstanding problem in this area is to find a distinguished representative for each birational equivalence class of such schemes which has, in some technical sense, the least complicated structure possible — this project is called the *minimal model program*.

For us, the relevance of such questions stems from the following observation: If we blow up an integral scheme at any proper closed subscheme, the structure map of the blowup will be birational (in the sense that it defines an invertible map in the rational category defined above — equivalently, it is an isomorphism over an open dense subscheme of the target). Indeed, it turns out that, at least in nice circumstances, *all* birational maps arise from sequences of blowups in an appropriate sense; hence, in practice, birational geometry is in large part the study of blowups of integral finite-type separated schemes.

Of particular importance, at least from the perspective of a singularity theorist, is the existence (in nice cases) of the birational maps called *resolutions of singularities*, originally due to Hironaka:

Theorem 1. *Let X be an integral separated scheme of finite type over a field k of characteristic zero. Then there exists a proper map $\pi : \tilde{X} \to X$ of k-schemes such that π is birational and $\tilde{X}$ is nonsingular.*

Indeed, Hironaka proved that this map can be taken to be the composition of the structure maps of a sequence of blowups with certain specified properties; we omit the details. In general, there is no particularly canonical choice of a resolution of singularities for a given X; nevertheless, many concepts in singularity theory can be defined by reference to the structure of a resolution of the singularity under consideration, provided one proves that the relevant statements are independent of the choices made. For example, the definitions of *rational singularities*, *Du Bois singularities*, *canonical singularities*, and their assorted generalizations and hangers-on, which provide various structural constraints on the failure of regularity, are of this flavor.

Intriguingly, the existence of such resolutions over fields of positive characteristic remains open in general. (At least to appearances — there are some latter-day claims by Hironaka and others to have resolved[1] the issue, but at present none of them seems to have been generally accepted as correct.) It is, however, known that the corresponding result for arbitrary schemes is false.

2.3 Regular Embeddings and Related Notions

In Lecture 5, we introduced the notion of a regular point of a locally Noetherian scheme — that is, one where the scheme locally "looks like a manifold" in the sense that the dimension of its Zariski tangent space is the same as its local dimension. Thinking of the tangent cone as giving the tangent directions within the Zariski tangent space which are actually realized, as discussed, we should expect in this case that the tangent cone will be equal to the entire Zariski tangent space; this can be shown by means of the following result, which we will not prove for now:

Proposition 2. *Let $X \hookrightarrow Y$ be a closed inclusion of schemes. Then, for each point $x \in X$, if we regard x also as a point of $C_X Y$ by identifying X with its image under the zero section, we have $\dim_x C_X Y = \dim_x Y$.*

The claimed equality at a regular point of a locally Noetherian scheme then follows because affine n-space over the residue field is integral and so has no proper closed subschemes of the same dimension. Conversely, if we are given a point in a locally Noetherian scheme such that the tangent cone is a linear fiber space over the residue field, Remark 1 will guarantee that in fact it is equal to the Zariski tangent space and so, by our dimensionality result, the scheme is regular at the point in question. In sum, we have the following result:

Proposition 3. *Let x be a point in a locally Noetherian scheme X. Then X is regular at x if and only if $C_x X \cong \mathbb{A}^n_x$ for some $n \geq 0$. In this case, $\dim_x X = n$ and $N_{x/X} = C_x X$.*

Thus we can express regularity in terms of the tangent cone. Since we have also developed more general notions of the Zariski normal scheme and the normal cone, though, we can also generalize our original question ("At which points does a scheme look like a manifold?") by giving a relative version: Which inclusions of schemes look like inclusions of manifolds? That is, if we have an inclusion of one possibly singular scheme into another, under what circumstances should we expect the relative behavior to be analogous to that of an embedding of manifolds?

This gives rise to the following definition:

Definition 6. *Let $i : X \hookrightarrow Y$ be a closed inclusion of locally Noetherian schemes and $k \geq 0$ an integer. We say that i is a* **regular embedding** *of codimension k if any of the following equivalent conditions holds:*

- *$C_X Y$ is a vector bundle of rank k.*
- *$\operatorname{codim}_Y X = k$ and $N_{X/Y}$ is a vector bundle of rank k.*

[1] Look, the important thing is that *I* think I'm funny.

- *(For those who have seen regular sequences:) For each point $x \in X$, the kernel of the natural surjection $\mathcal{O}_{Y,x} \twoheadrightarrow \mathcal{O}_{X,x}$ is generated by a regular sequence of length k.*

(From the conditions, it is again clear that in this circumstance we will have $N_{X/Y} = C_X Y$.)

This is to say that, just as regular points are those at which the tangent behavior matches that of manifolds in the sense that the tangent space has the correct dimension and all tangent directions are actually attained, regular embeddings are those such that the normal behavior matches that of an embedding of manifolds — the Zariski normal scheme is a vector bundle of the correct rank, and all of these normal directions are actually attained.

Many algebro-geometric constructions — e.g., the theories of *canonical bundles* of schemes and *characteristic classes* of vector bundles — deal primarily with nonsingular schemes, at least classically; similarly, there are some concepts which apply most readily to regular embeddings. *Intersection theory*, for example, concerns itself with enumerative questions like the problem of Apollonius mentioned at the outset of Lecture 1; the approach requires a robust notion of the "multiplicity with which two subschemes of some ambient scheme intersect each other" (creatively called their *intersection multiplicity*), and this turns out to be most feasible in the situation where at least one of them is regularly embedded. Essentially, one can think that a good notion of intersection multiplicity should be defined so as to respect small perturbations in some sense, and in order for the notion of "small perturbations" of a given subscheme to be well-behaved it is necessary that the directions normal to it in the ambient scheme have a manageable structure as well.

Although they are more typically stated in algebraic language, in terms of the aforementioned *regular sequences*, the following notions — which are extremely important and commonly used in algebraic geometry — are essentially concerned with regular embeddings, at least in the local Noetherian setting where they are most often considered:

Definition 7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and set $Y := \operatorname{Spec} R$. Then, for any closed subscheme X of Y, we define the* **depth** *of X in Y to be*

$$\operatorname{depth}_Y X := \max\{\operatorname{codim}_Y X' \mid X' \supseteq X \text{ a regularly-embedded closed subscheme of } Y\}.$$

If $\operatorname{depth}_Y X = \operatorname{codim}_Y X$ for every such X, we say that R is a **Cohen-Macaulay ring***; more generally, we say a locally Noetherian scheme is* **Cohen-Macaulay** *if the local ring at every point is Cohen-Macaulay.*

It turns out that every nonsingular scheme is Cohen-Macaulay, and indeed that Cohen-Macaulay schemes retain the properties of nonsingular schemes needed for many concrete applications; hence the class of such schemes gives a reasonable setting where intuitions from differential geometry often remain applicable without being so stringent as to exclude the possibility of singularities entirely.

Relatedly, attempts to homologically measure the failure of regularity (again, typically framed more in terms of sequences than embeddings) lead to the theory of *Koszul complexes*, and this in turn provides one entry point for the study of *local cohomology*, an algebro-geometric analogue to certain relative cohomology groups appearing in algebraic topology. Such considerations are far beyond our present scope, but yield the following noteworthy interpretation of Cohen-Macaulay schemes: These are the algebro-geometric analogues of the locally conical spaces whose links have cohomology concentrated in a single degree, a generalization of the class of homology manifolds. (Depending where you're coming from, this may or may not mean anything to you — either way is fine.)

We conclude our discussion of regular embeddings with an application more pertinent to what we have learned thus far — namely, the universal property of the blowup. Given a closed subscheme X of a locally Noetherian scheme Y, we obtain the blowup $\mathrm{Bl}_X Y$ essentially by separating all of the limiting normal directions to X in Y at each point; hence, we should expect that there is a unique normal direction at each point of the closed subscheme $E := X \times_Y \mathrm{Bl}_X Y \cong \mathbb{P}(C_X Y)$ of the blowup given by pulling back X. Indeed, this is true: The closed inclusion $E \hookrightarrow \mathrm{Bl}_X Y$ is a regular embedding of codimension 1, which is to say exactly that its normal cone $C_E \mathrm{Bl}_X Y$ is a line bundle, or equivalently that the radial cone $R_E \mathrm{Bl}_X Y$ is. (For this latter reason, E can be realized as the vanishing locus of the natural section $\mathbb{A}^1_{\mathrm{Bl}_X Y} \cong \mathrm{Spec}_+ \mathcal{O}_{\mathrm{Bl}_X Y} \to R_E \mathrm{Bl}_X Y$, hence of the corresponding linear form on the dual line bundle, and so the "exceptional divisor" E is indeed a divisor.) As one might expect, the structure map of the blowup is universal among maps with this property:

Proposition 4 (universal property of the blowup)**.** *Let Y be a locally Noetherian scheme and $X \hookrightarrow Y$ a closed inclusion. Then, for any locally Noetherian scheme Z and map $\phi : Z \to Y$ such that the closed inclusion $X \times_Y Z \hookrightarrow Z$ is a regular embedding of codimension 1, ϕ factors uniquely through the structure map of the blowup of Y at X:*

$$\begin{tikzcd}
X \times_Y Z \arrow[r, hook, "\text{regular}", "\text{codim } 1"'] \arrow[d] & Z \arrow[r, dashed, "\exists!"] \arrow[d, "\phi"] & \mathrm{Bl}_X Y \arrow[dl] \\
X \arrow[r, hook] & Y &
\end{tikzcd}$$

(An appropriate modification of this result will remain true without the Noetherianity hypotheses, but we won't get into it.)

3 A Glimpse of Flatness

We can also use the normal cone to make sense of the following important and ubiquitous definition, originally due to Serre:

Definition 8. *A ring map $R \to S$ is said to be* **flat** *if $- \otimes_R S$ is an exact functor on the category of R-modules — that is, for any short exact sequence*

$$0 \to A \to B \to C \to 0$$

of R-modules,

$$0 \to A \otimes_R S \to B \otimes_R S \to C \otimes_R S \to 0$$

remains exact. (Recall that exactness on the right is automatic, so the key requirement is that $A \otimes_R S \to B \otimes_R S$ remains injective.)

A map $\phi : X \to Y$ of schemes is called **flat** *if any of the following equivalent conditions holds:*

- *There are compatible affine open covers of X and Y such that the induced ring maps are all flat.*
- *For every pair of affine opens* $\operatorname{Spec} R \subseteq Y$ *and* $\operatorname{Spec} S \subseteq \phi^{-1}(\operatorname{Spec} R)$*, the induced ring map $R \to S$ is flat.*
- *For every point $x \in X$, the induced map $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ of ring maps is flat.*
- *The pullback functor ϕ^* from the category of quasicoherent sheaves on Y to the category of quasicoherent sheaves on X is exact.*

From an algebraic perspective, the advantages of working with maps of this type are clear; geometrically, however, their significance is not immediately obvious. To make sense of it, we first note the following:

Proposition 5. *Let $\phi : X \to Y$ be a map of schemes and $Z \hookrightarrow Y$ a closed inclusion. Then there is a natural induced map from the radial cone of $\phi^{-1}(Z)$ in X to the radial cone of Z in Y commuting with the natural projections:*

$$\begin{array}{ccc} R_{\phi^{-1}(Z)}X & \longrightarrow & R_Z Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & Y \end{array}$$

Moreover, the corresponding map $R_{\phi^{-1}(Z)}X \to X \times_Y R_Z Y$ is a closed inclusion of conical fiber spaces which is an isomorphism over $X \setminus \phi^{-1}(Z)$.

Proof sketch. We verify this in the case where X and Y are both affine. Then ϕ is given by a ring map $f : R \to S$ and Z is cut out in $Y = \operatorname{Spec} R$ by some ideal $I \subseteq R$, while $\phi^{-1}(Z)$ is cut out in $X = \operatorname{Spec} S$ by $J := IS$, so that the blowup algebras of Z in Y and $\phi^{-1}(Z)$ in X are given by $R[It] \subseteq R[t]$ and $S[Jt] \subseteq S[t]$ respectively. Hence the claimed map on radial cones is given by the natural extension $R[It] \to S[Jt]$ of f which takes each $at \in It$ to the corresponding element $f(a)t \in Jt$.

To see the claim that the map to the fiber product is a closed inclusion, it is enough to note that $S \otimes_R R[It] \to S[Jt]$ is a surjection by virtue of the fact that any set of generators for I will be sent by f to a set of generators for $J = IS$. Finally, the claim about the restriction over the complement of $\phi^{-1}(Z)$ holds since, for any $g \in I$, the restriction of the map on radial cones over $\operatorname{Spec} R_g \subseteq \operatorname{Spec} R$ is given by the usual map $R_g[t] \to S_{f(g)}[t]$ induced by f and hence we obtain the identity map $S_{f(g)}[t] \to S_{f(g)}[t]$ after factoring through the tensor product. □

Thus, in particular, we find at each point z of the pullback $\phi^{-1}(Z)$ of our closed subscheme Z of Y to X that the union $C_{\phi^{-1}(Z)}X|_z$ of the normal directions to $\phi^{-1}(Z)$ is a closed subscheme of the union $C_Z Y|_{\phi(z)}$ of the normal directions to Z in Y at the corresponding point of Z. Note also that our map $R_{\phi^{-1}(Z)}X \to R_Z Y$ behaves well with respect to projectivization and that the map $\operatorname{Bl}_{\phi^{-1}(Z)} X \to \operatorname{Bl}_Z Y$ thus induced is precisely the one given by applying the universal property of $\operatorname{Bl}_Z Y \to Y$ given in Proposition 4 to the natural map $\operatorname{Bl}_{\phi^{-1}(Z)} X \to Y$.

We are now ready to give a geometric interpretation of flatness, at least in the locally Noetherian case:

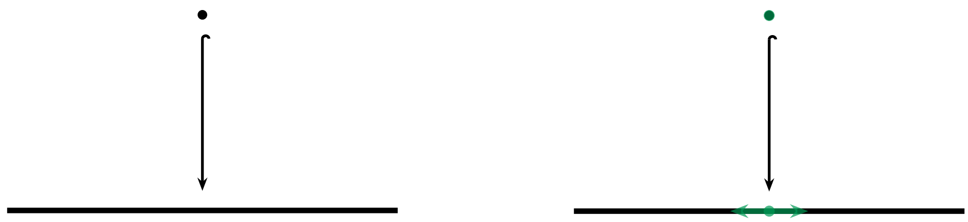

Figure 7: The (non-flat) inclusion of a point into a line discussed in Example 5; in the version on the right, the tangent cone to the line at the origin and the normal cone of the fiber over this point are superimposed in green.

Proposition 6. *Let $\phi : X \to Y$ be a map of locally Noetherian schemes. Then ϕ is flat if and only if, for every point $y \in Y$, the natural closed inclusion $C_{\phi^{-1}(y)}X \hookrightarrow \phi^{-1}(y) \times_y C_y Y$ is an isomorphism.*

(If y is a non-closed point, we make sense of these expressions by working in $\operatorname{Spec} \mathcal{O}_{Y,y}$ instead of Y and in $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}$ instead of X, so that y becomes closed; for points which are already closed, this replacement does not affect the resulting normal cones.)

Thus to say that a map is flat is to say that, at every point of the source, “the normal directions to the fiber it is contained in are precisely the tangent directions to the target at the point it is mapped to”. (As usual, the subtleties arising from non-reduced behavior compel us to maintain a bit of ironic distance from this statement.)

As a basic non-example, we have a typical closed inclusion:

Example 5. *The inclusion $\mathbb{A}^0_{\mathbb{C}} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$ of the origin into the affine line over $\mathbb{C}$ is not flat. Geometrically, we can see this by observing that the tangent cone to $\mathbb{A}^1_{\mathbb{C}}$ at the origin (or any other closed point) is itself an affine line, while the normal cone to a single point in itself is again a point; this situation is depicted in Figure 7. Algebraically, this map corresponds to the map of $\mathbb{C}$-algebras $\mathbb{C}[t] \to \mathbb{C}$ given by $t \mapsto 0$; the tangent cone at the origin in the target is then given by $\operatorname{Spec} \operatorname{gr}_{(t)} \mathbb{C}[t] = \operatorname{Spec} \mathbb{C}[e]$ for e the equivalence class of t in $(t)/(t)^2$, while the normal cone to the fiber over this point in the source is $\operatorname{Spec} \operatorname{gr}_{(0)} \mathbb{C} = \operatorname{Spec} \mathbb{C}$; we can then see that the induced map $\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}[e]) \to \operatorname{Spec} \mathbb{C}$ is not an isomorphism.*

On the other hand, *every* open inclusion is flat; we give one example here to illustrate the intuition.

Example 6. *The inclusion $\mathbb{A}^1_{\mathbb{C}} \setminus \mathbb{A}^0_{\mathbb{C}} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$ of the complement of the origin into the affine line over $\mathbb{C}$ is flat. If we consider a closed point p of the line other than the origin, we can see that the inclusion is locally an isomorphism near p, so the normal cone to the fiber is simply the tangent cone again. Algebraically, our map is given by $\mathbb{C}[t] \to \mathbb{C}[t]_t$ and such a point is cut out by the ideal $(t-a)$ of $\mathbb{C}[t]$ for $a \in \mathbb{C}^*$; the induced map on tangent cones is then given by the identity map from $\operatorname{gr}_{(t-a)} \mathbb{C}[t] = \mathbb{C}[e]$ to $\operatorname{gr}_{(t-a)} \mathbb{C}[t]_t = \mathbb{C}[e]$, where e is the equivalence class of $t-a$ in $(t-a)/(t-a)^2$ in each case. This is illustrated in Figure 8.*

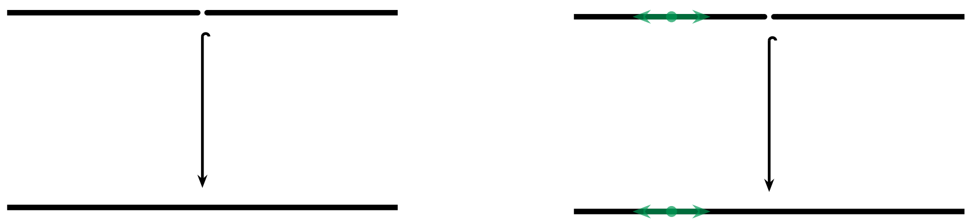

Figure 8: The (flat) inclusion of an open subset of the affine line discussed in Example 6; in the version on the right, the tangent cone to the line at a typical point and the normal cone of the fiber over this point are superimposed in green.

Over the origin o, on the other hand, the situation is as follows. The corresponding fiber of our map is now empty, so its normal cone in the domain is $C_\emptyset(\mathbb{A}^1_\mathbb{C} \setminus \mathbb{A}^0_\mathbb{C}) = \emptyset$. Now, the tangent cone to the affine line itself at o is another affine line $C_o\mathbb{A}^1_\mathbb{C} \cong \mathbb{A}^1_\mathbb{C}$; we must verify the equality of the fiber's normal cone with the product $\emptyset \times_o C_o\mathbb{A}^1_\mathbb{C}$ of the fiber with this tangent cone. Since the fiber product of anything with the empty scheme is empty, this equality holds. Algebraically, since o is cut out by the ideal (t), this is just to say that $\operatorname{gr}_{(t)} \mathbb{C}[t]_t = \operatorname{gr}_{(1)} \mathbb{C}[t]_t$ and $\mathbb{C}[t]_t/(t)\mathbb{C}[t]_t \otimes_{\mathbb{C}[t]/(t)} \operatorname{gr}_{(t)} \mathbb{C}[t] = \mathbb{C}[t]_t/(1) \otimes_\mathbb{C} \operatorname{gr}_{(t)} \mathbb{C}[t]$ are both zero.

Finally, our map becomes an isomorphism if we localize at the generic point, so we can see that here we have an isomorphism of tangent cones as well.

Our first two examples have been inclusions, so that the fiber over a given point of the domain is either empty or again a point; of course, we can also consider maps with more exciting fibers:

Example 7. *The standard projection $\pi : \mathbb{A}^2_\mathbb{C} \to \mathbb{A}^1_\mathbb{C}$ to the first coordinate is flat. Algebraically, this map is given by the usual inclusion $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, y]$; each closed point p of $\mathbb{A}_1$ will then be cut out by the ideal $(x - a)$ for some $a \in \mathbb{C}$. As usual, the tangent cone to $\mathbb{A}^1_\mathbb{C}$ at this point is then given by $\operatorname{Spec} \operatorname{gr}_{(x-a)} \mathbb{C}[x] = \operatorname{Spec} \mathbb{C}[e]$ for e the equivalence class of $x - a$ in $(x - a)/(x - a)^2$; likewise, the normal cone to the fiber over p in $\mathbb{A}^2_\mathbb{C}$ is $\operatorname{Spec} \operatorname{gr}_{(x-a)} \mathbb{C}[x, y] = \operatorname{Spec}((\mathbb{C}[y])[e])$, and the induced map $C_{\pi^{-1}(p)}\mathbb{A}^2_\mathbb{C} \to C_p\mathbb{A}^1_\mathbb{C}$ is given by the $\mathbb{C}$-algebra map $\mathbb{C}[e] \to (\mathbb{C}[y])[e] \cong \mathbb{C}[e, y]$ taking e to e. Since the map $\pi^{-1}(p) \to p$ is the one corresponding to the usual map $\mathbb{C} \to \mathbb{C}[y]$ and the projection $C_{\pi^{-1}(p)}\mathbb{A}^2_\mathbb{C} \to \pi^{-1}(p)$ corresponds to the inclusion $\mathbb{C}[y] \hookrightarrow \mathbb{C}[e, y]$, we can then see that $C_{\pi^{-1}(p)}\mathbb{A}^2_\mathbb{C} \to \pi^{-1}(p) \times_p C_p\mathbb{A}^1_\mathbb{C}$ is given algebraically by the $\mathbb{C}$-algebra map $\mathbb{C}[y] \otimes_\mathbb{C} \mathbb{C}[e] \cong \mathbb{C}[e, y] \to \mathbb{C}[e, y]$ taking e to e and y to y. This is indeed an isomorphism, so the normal cone to the fiber has the desired product structure. This situation is illustrated in Figure 9.*

If we let η be the generic point of $\mathbb{A}^1_\mathbb{C}$, we have $\operatorname{Spec} \mathcal{O}_{\mathbb{A}^1_\mathbb{C},\eta} = \operatorname{Spec} \mathbb{C}(x) = \eta$ and $\mathbb{A}^2_\mathbb{C} \times_{\mathbb{A}^1_\mathbb{C}} \operatorname{Spec} \mathcal{O}_{\mathbb{A}^1_\mathbb{C},\eta} \cong \operatorname{Spec} \mathbb{C}(x)[y]$; since the whole domain of the map from the latter to the former induced by π is a single fiber, the normal cones involved are trivial and so the product condition follows automatically.

Example 8. *Let $Y := \mathbb{A}^1_\mathbb{C}$ be the affine line and $X := V(x(y - x^2)) \subseteq \mathbb{A}^2_\mathbb{C}$ the union of the y-axis with a parabola in the affine plane. Let $\phi : X \to Y$ be the restriction of the usual*

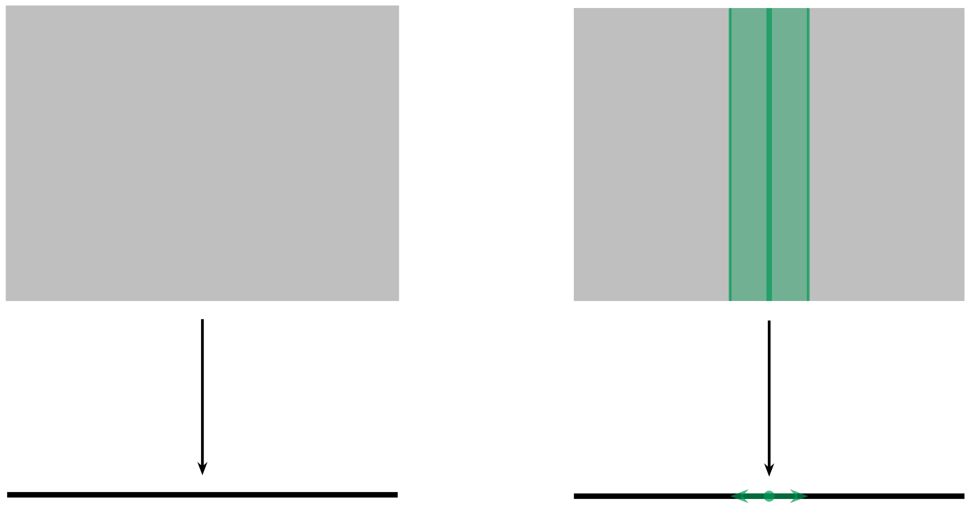

Figure 9: The (flat) projection of the affine plane to the affine line discussed in Example 7; in the version on the right, the tangent cone to the line at a typical point and the normal cone of the fiber over this point are superimposed in green.

projection onto the first coordinate, so that the corresponding ring map is the standard map $\mathbb{C}[x] \to \mathbb{C}[x,y]/(x(y-x^2))$. *Then* ϕ *is not flat.*

If p *is a closed point of* Y *other than the origin, this fact is not evident; indeed, letting* $(x-a)$ *for* $a \in \mathbb{C}^*$ *be the ideal cutting out* p *as in our prior examples, we have the usual isomorphism* $C_p\mathbb{A}^1_{\mathbb{C}} \cong \operatorname{Spec}\mathbb{C}[e]$, *while* $\phi^{-1}(p) = \operatorname{Spec}\mathbb{C}[x,y]//(x(y-x^2), x-a) \cong \operatorname{Spec}\mathbb{C}[y]/(a/(y-a^2)) \cong \operatorname{Spec}\mathbb{C}$ *and* $C_{\phi^{-1}(p)}X = \operatorname{Spec}\operatorname{gr}_{(x-a)}\mathbb{C}[x,y]/(x(y-x^2)) \cong \operatorname{Spec}\mathbb{C}[e,y]/(a(y-a^2)) \cong \operatorname{Spec}\mathbb{C}[e]$. *Hence, as in Example 6,* $C_{\phi^{-1}(p)}X \cong \phi^{-1}(p) \times_p C_pY$, *so* p *does not witness the failure of flatness.*

However, if we let $o \in Y$ *be the origin, then we again have* $C_oY \cong \operatorname{Spec}\mathbb{C}[e]$, *but* $\phi^{-1}(o) = \operatorname{Spec}\mathbb{C}[x,y]/(x(y-x^2), x) \cong \operatorname{Spec}\mathbb{C}[y]$ *and* $C_{\phi^{-1}(o)}X = \operatorname{Spec}\operatorname{gr}_{(x)}\mathbb{C}[x,y]/(x(y-x^2)) \cong \operatorname{Spec}\mathbb{C}[e,y]/(ey)$. *Thus the closed inclusion* $C_{\phi^{-1}(o)}X \hookrightarrow \phi^{-1}(o) \times_o C_oY$ *is given by the standard quotient map from* $\mathbb{C}[y] \otimes_{\mathbb{C}} \mathbb{C}[e] \cong \mathbb{C}[e,y]$ *to* $\mathbb{C}[e,y]/(ey)$, *so the normal cone to the fiber does not have the desired product structure; instead, we see that there is a normal direction corresponding to the tangent direction in the target only at the point where the vertical line meets the parabola, and this direction is missing at every other point of the fiber. (A technical note: This way of presenting things might tempt one to say that this map is "flat at the point where the parabola meets the line", but this is not true; the upshot is that, even locally around this point, the normal cone to the fiber does not have the correct product structure, even though we have the requisite normal direction at the point itself.)*

These situations are illustrated in Figure 10.

This concludes our foray into the geometric intuitions for flatness. It should be noted that this is far from a complete treatment of the subject; in keeping with the theme of the lecture, we have mainly been concerned with introducing the basics of the relationship

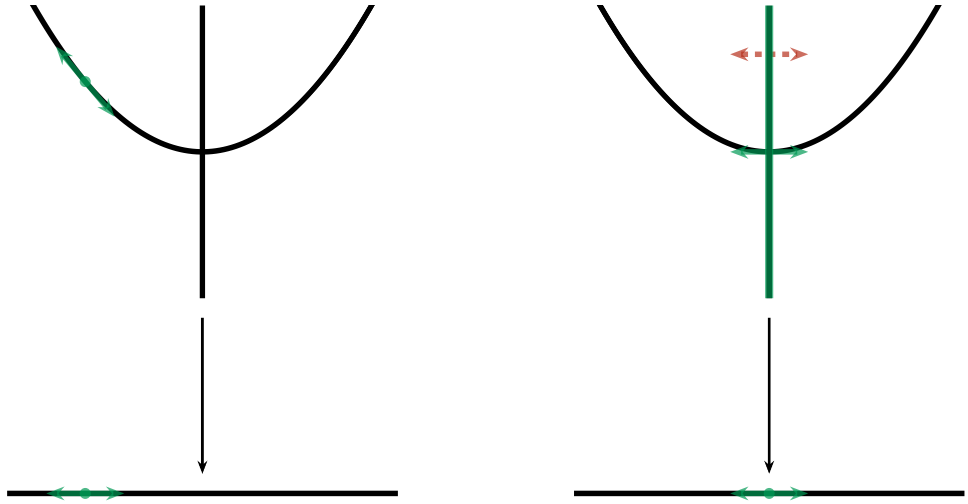

Figure 10: The (non-flat) map ϕ of Example 8. In both versions, we superimpose the tangent cone to the line at a chosen point and the normal cone of the fiber over this point in green; on the left, the chosen point is an arbitrary closed point away from the origin, and on the right the point is the origin itself. In the version on the right, we also indicate one of the "missing normal directions" not included in the normal cone in red.

between flatness and the normal cone construction, rather than giving a comprehensive picture.

4 Some Applications of Flatness

We conclude by surveying a few of the uses of flatness in algebraic geometry. As in Section 2, our goal is only to gesture in the direction of more advanced topics for the interested student, not to cover them with any particular depth or rigor.

4.1 Moduli Spaces and Deformations

In Lectures 12 and 13, we introduced projectivizations of conical fiber spaces — for example, the projective space $\mathbb{P}^n_{\mathbb{C}}$. As discussed, this is "the space of lines through the origin in $\mathbb{A}^{n+1}_{\mathbb{C}}$" and indeed we showed that each of its closed points corresponds to some such line. However, there is one issue which we did not explicitly tackle: Why is this "*the* space" of such lines? For example, we could very easily construct a different scheme with points corresponding to our lines simply by taking a disjoint union:

$$\coprod_{\substack{\ell \text{ a line through} \\ \text{the origin in } \mathbb{A}^{n+1}_{\mathbb{C}}}} \operatorname{Spec} \mathbb{C}.$$

Of course, this is a bit silly, and from the quotient construction of projective space it intuitively seems like the "more correct" way to define our space of lines, but the question remains: How should we characterize this intuition mathematically, and can we do so in a way that applies to other situations where we want to define "the space of ⟨algebro-geometric objects of some fixed type⟩"?

Clearly, we want some limitation on how the different points in our space — that is, different objects of the given type — fit together. As such, we should require not just that points of our space — i.e., inclusions of a field spectrum into the space — correspond to objects of the sort we are interested in, but that more general maps to the space correspond to "families of objects which vary nicely"; for example, in the case of our projective space, we would like to say that map from the spectrum of a discrete valuation ring which sends the closed point to some $\ell \in \mathbb{P}^n_{\mathbb{C}}$ corresponds somehow to a "germ of a 1-parameter family of lines through the origin converging to the line corresponding to ℓ" and so forth.

It turns out that the best way to capture these concepts is, very roughly, as follows. Given a scheme P, we can get a notion of "a collection of objects of the sort we are considering indexed by P" by taking scheme maps $\phi : \mathfrak{X} \to P$ such that, for each point $p \in P$, the fiber $\phi^{-1}(p)$ is an object of the desired type. (Assuming our objects are schemes of some sort, at least.) To exclude pathologies (such as, e.g., $\mathfrak{X}$ being a disjoint union of the fibers of ϕ) and ensure that such a map defines a "nicely-varying family", we require as a baseline that it be flat; the sense in which the fibers "fit together nicely" is then given by Proposition 6.

For a given type of algebro-geometric object, such a map $\mathfrak{X} \to P$ is then called a *(flat) family* of ⟨objects of the given type⟩ over P, or, particularly in cases where P comes with some distinguished point (like the closed point of a DVR spectrum), a *deformation* of the "central fiber" over this distinguished point. The study of deformations of objects of some fixed type, particularly over a fixed base or collection of bases, is called the *deformation theory* of objects of that type. (I'm being substantially over-general and loose here, necessarily — depending on the details of the "type of object" we are considering and context we are considering it in, the exact meanings of terms will vary in practice.)

This gives us a rubric for making precise the notion of "the space of ⟨objects of a given type⟩". Specifically, this should be space M together with a natural bijection

$$\left\{\substack{\text{scheme maps}\\ P\to M}\right\} \overset{\sim}{\longleftrightarrow} \left\{\substack{\text{flat families of}\\ \langle\text{objects of the type}\rangle\text{ over }P}\right\}$$

as P runs over all schemes. (Again, this is just a sketch — depending on context, we may restrict our attention to certain subcategories of the category of schemes, or even expand to larger settings like the category of *stacks*[2] mentioned in Lecture 12.) If such an M exists, it is called the *(fine) moduli space* of ⟨objects of the given type⟩. Thus, for example, the "moduli space of lines through the origin in $\mathbb{A}^{n+1}_{\mathbb{C}}$" is $\mathbb{P}^n_{\mathbb{C}}$, and not the disjoint union we defined earlier, precisely because the former can account for non-constant families of lines in this way and the latter cannot. Indeed, for a given map $P \to \mathbb{P}^n_{\mathbb{C}}$, we can obtain the corresponding family of lines through the origin by pulling back the tautological bundle $\mathbb{T}(\mathbb{A}^{n+1}_{\mathbb{C}}) \to \mathbb{P}^n_{\mathbb{C}}$ (together with some embedding data for its fibers, strictly speaking) over P. More generally, if M is a moduli space in this sense, the identity map $M \to M$ is identified under the natural

[2] Or, to be really precise, I should say something like "one of the categor*ies* of stacks"; the details are outside our scope here.

bijection with a *universal family* $U \to M$ of objects of the type in question, and naturality then guarantees that the family corresponding to $P \to M$ is given by pulling back U over P.

Remark 3. *Strictly speaking, our discussion of the way $\mathbb{P}^n_{\mathbb{C}}$ functions as a moduli space has been a bit imprecise; although it is true that the closed points of this space correspond to lines through the origin in $\mathbb{A}^{n+1}_{\mathbb{C}}$, I've been talking as though this correspondence extended to every point, which doesn't really make sense. Likewise, the definition of a "family of objects of a certain type" being a morphism where each fiber is such an object may be problematic in practice, depending on how carefully we formulate our definition of the class of objects under consideration. However, as mentioned, it is difficult to be truly precise at this level of generality, and so I've focused mostly on giving you the flavor of things.*

In the case of $\mathbb{P}^n_{\mathbb{C}}$ (and other projective spaces), the mathematically rigorous version of our discussion here is essentially captured by Theorem 3 of last week's lecture.

There are many different subfields of moduli theory, of interest to different types of algebraic geometers, and we will make no attempt at an exhaustive list. Some of the most famous and classical moduli spaces include the *Grassmannians*, which are moduli spaces of fixed-dimensional vector subspaces of a given vector space (the projective spaces $\mathbb{P}^n_k$ we have seen thus far comprise the special case where the chosen subspace dimension is 1), the moduli spaces of various sorts of projective curves (e.g., elliptic curves — although in this case the moduli space does not actually exist in the world of schemes), and the *Hilbert schemes*, which are moduli spaces of closed subschemes of given schemes.

4.2 Smooth Maps and Related Concepts

In Lecture 10, we discussed the notion of a smooth scheme over a given ground field. Compared to most of the definitions we've seen in this course, the circumstances in which this concept can be applied are a bit restrictive; typically, we've been more interested in working with general, relative versions of concepts which can be applied to any morphism of schemes than in focusing in on the setting of schemes over a field specifically. As you may have suspected, such a generalization exists in this case as well; we've avoided it thus far only because one of the ingredients involved is flatness. For simplicity, we retain a local Noetherianity hypothesis on the target:

Definition 9. *Let $\phi : X \to Y$ be a map of schemes, and suppose that Y is locally Noetherian. Then, for $n \geq 0$ an integer, we say that ϕ is* **smooth** *of relative dimension n if it is locally of finite type and flat, it has pure relative dimension n (i.e., every fiber is purely n-dimensional), and the relative tangent scheme $T_{X/Y}$ is a vector bundle of rank n on X.*

Note that, in the case where Y is the spectrum of a field, we retrieve our original definition of smoothness — the flatness hypothesis does nothing in this case by Proposition 6 since all of X is a single fiber and so the normal cone to this fiber is simply the fiber itself (this is a geometric rephrasing of the algebraic fact that everything is flat over a field). In particular, since the properties defining smoothness are all preserved under pullback, every fiber of a smooth map of relative dimension n will then be a smooth scheme of dimension n over the residue field at the chosen point of the target.

Intuitively, smooth maps are the algebro-geometric analogues to submersions of smooth manifolds — note that, in the differential-geometric setting, the appropriate analogue to flatness is satisfied since the normal bundle to each fiber is trivial and of the same rank as the tangent space at the corresponding point of the target, which perhaps gives some indication as to why the flatness hypothesis is needed.

The following special case is of independent interest:

Definition 10. *A map of schemes is called* **étale** *if it is smooth of relative dimension 0.*

On the differential geometry side, these correspond to smooth maps which are isomorphisms locally on their domains — that is, those which are covering spaces in the sense of algebraic topology. This identification turns out to be the key to defining algebro-geometric analogues to several other classical concepts; using the typical correspondence between covering spaces and subgroups of the fundamental group, for example, we can consider the (finite) étale maps to a given (locally Noetherian) scheme to reverse-engineer a sort of scheme-theoretic fundamental group, called the *étale fundamental group*. With a bit more theory, it is also possible to use étale maps to construct a notion of cohomology, called *étale cohomology*, which retrieves the ordinary singular cohomology groups of the *classical* topology on finite-type $\mathbb{C}$-schemes in nice cases, and therefore provides an analogue to these groups for schemes defined over other fields. (The process of passing from a finite-type $\mathbb{C}$-scheme to the corresponding space endowed with the classical topology and a sheaf of "holomorphic functions" is called *analytification*; essentially, the point is that, in each open chart where our scheme is a subspace of $\mathbb{A}^n_{\mathbb{C}}$, we can consider the corresponding subspace of the classical space $\mathbb{C}^n$.)