An Introduction to the Deformation Theory of Complete Intersection Singularities

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Brief introductory note because I got lazy: analytic spaces

- A finite-type $\mathbb{C}$-scheme is a locally ringed space which is locally isomorphic to the vanishing of some ideal of $\mathbb{A}^n_{\mathbb{C}}$, where $\mathbb{A}^n_{\mathbb{C}}$ has the Zariski topology and $\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}$ is the sheaf of polynomials.

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- We'll use the latter setup in this talk because there were things I didn't get around to verifying in the algebraic setting; if it makes you uncomfortable and you like to live dangerously you can try ignoring it and just thinking about polynomials.

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- We can now ask questions about how the fibers relate to each other, what kind of deformations of a certain space are possible, etc.

Our situation

- We'll look at map germs

$$F : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0);$$

  these are given by a choice of k coordinate functions defined and holomorphic in a neighborhood of the origin in $\mathbb{C}^{n+k}$. For convenience, we'll write X for the source and S for the target.

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- We'll call F a **deformation of the complete intersection singularity** $(X_0, 0)$.

Aside: The Milnor fiber

- The general fiber of F, which is smooth, is called its **Milnor fiber**.
- Some people care about how its topology relates to the singularities of $(X_0, 0)$.
- A simpler question: Is it obvious whether it's even an invariant of $(X_0, 0)$? That is, could it depend on F?

Is there a universal deformation?

- If $F : (X, 0) \to (S, 0)$ is a deformation of $(X_0, 0)$ and $g : (S', 0) \to (S, 0)$ (for $S' = \mathbb{C}^{k'}$) is a nice enough map, we get another deformation via pullback:

$$\begin{tikzcd} (X', 0) \arrow[r, "\tilde{g}"] \arrow[d, "F'"'] & (X, 0) \arrow[d, "F"'] \\ (S', 0) \arrow[r, "g"] & (S, 0) \end{tikzcd}$$

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- We can ask whether there is a deformation of $(X_0,0)$ universal in the sense that every other deformation can be written uniquely as a pullback from it.
- Answer: Only if $(X_0,0)$ is smooth (i.e., **extremely boring**).

Is there a versal deformation?

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- Being **versal** is like being universal without the uniqueness — we ask only if there's a deformation of $(X_0, 0)$ such that we can get all others as pullbacks, maybe non-uniquely.
- This still seems like a pretty miraculous thing to ask for, but it turns out that such deformations do exist if $(X_0, 0)$ has only an isolated singularity at the origin (i.e., $X_0 \setminus \{0\}$ is smooth — this implies that all nearby fibers have at worst isolated singularities as well).

Preliminary stuff about vector fields

- Let $\mathcal{T}_{X,0}$ denote the $\mathcal{O}_{X,0}$-module of germs of holomorphic vector fields on X (and define $\mathcal{T}_{S,0}$ similarly). Concretely, $\mathcal{T}_{X,0} = \mathcal{O}_{X,0}\frac{\partial}{\partial x_1} \oplus \ldots \oplus \mathcal{O}_{X,0}\frac{\partial}{\partial x_{n+k}}$.

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- Then $F^*\mathcal{T}_{S,0} = \mathcal{O}_{X,0} \otimes_{\mathcal{O}_{S,0}} \mathcal{T}_{S,0} = \mathcal{O}_{X,0}\frac{\partial}{\partial t_1} \oplus \ldots \oplus \mathcal{O}_{X,0}\frac{\partial}{\partial t_k}$ corresponds to a (holomorphic) choice at each point of X of a tangent vector to S.

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- We can view the derivative of F as a map $dF : \mathcal{T}_{X,0} \to F^*\mathcal{T}_{S,0}$.

Toward the Kodaira-Spencer map

- Set $T^1_{X/S,0} = \mathrm{coker}(dF)$ $(= \mathrm{Ext}^1_{\mathcal{O}_{X,0}}(\Omega_{X/S,0}, \mathcal{O}_{X,0}))$ and $T^1_{X_0,0} = T^1_{X/S,0}/(F_1, \ldots, F_k)T^1_{X/S,0}$ $(= \mathrm{Ext}^1_{\mathcal{O}_{X_0,0}}(\Omega_{X_0,0}, \mathcal{O}_{X_0,0}))$.

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- Intuition: $T^1_{X/S,0}$ gives us a choice for each point in X of a way to move in S that will actually deform X at S in a nontrivial way (this is why we mod out by dF). $T^1_{X_0,0}$ is just the restriction to X_0.

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- Since $T^1_{X_0,0}$ is independent of F, we can think of it as "the vector space of all possible ways to deform X_0".

The Kodaira-Spencer map

- The natural map

$$\rho_{F,0} : \mathcal{T}_{S,0} \to T^1_{X/S,0}$$

is called the **Kodaira-Spencer map**. Its restriction

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- Fact: A deformation of $(X_0, 0)$ is versal if and only if its (reduced) Kodaira-Spencer map is surjective.

Existence of versal deformations in the isolated case

- Notice that $T^1_{X/S,0} = \mathrm{coker}(dF)$, by definition, is supported only on the critical locus of F (i.e., the locus where the Jacobian drops rank and hence dF is not surjective).

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- Hence $T^1_{X_0,0}$ is supported only on the singular locus of X_0, so it's finite-dimensional if X_0 has an isolated singularity at the origin.
- We can use this to extend F in a straightforward way and get a versal deformation.

Example: a cusp

- Let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be given by $F(x, y) = y^2 - x^3$.

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- Now we can build a versal deformation $\tilde{F} : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ by $F(x, y, u) = (y^2 - x^3 + ux, u)$.

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Check out Looijenga's *Isolated Singular Points on Complete Intersections*!