Embrace the Singularity: An Introduction to Stratified Morse Theory

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April 15, 2020

Introduction

- Outline:
  - Refresher on Classical Morse Theory
  - Structure of Singular Varieties
  - Stratified Morse Theory
- References:
  - Milnor, *Morse Theory*
  - Goresky and MacPherson, *Stratified Morse Theory*

Quick Reminder: Analytification

- A complex quasi-affine variety $X \subseteq \mathbb{C}^n$ can be considered with the classical topology on $\mathbb{C}^n$.
- An abstract variety over $\mathbb{C}$ is locally quasi-affine, so we can "analytify" it as well.
- X is smooth over $\mathbb{C}$ if and only if its analytification is a manifold.

Setup and Definitions

- Let X be a compact smooth n-manifold and $f : X \to \mathbb{R}$ a smooth function.
- For $a \in \mathbb{R}$, let $X_{\leq a}$ denote the subset $f^{-1}((-\infty, a])$.

Definition

A point $x \in X$ is called a **critical point** *of f if the differential $df_x : T_x X \to T_{f(x)}\mathbb{R} \cong \mathbb{R}$ is zero. For any such point, the value $f(x) \in \mathbb{R}$ is called a* **critical value** *of f.*

Classical Morse Theorem, Part A

Theorem (Fundamental Theorem of Morse Theory, Part A)

If the interval $[a, b]$ contains no critical values of f, then $X_{\leq a}$ and $X_{\leq b}$ are homeomorphic.

Morse Functions and Morse Index

Definition

- *The* **Hessian** *of f at a point $x \in X$ is the $n \times n$ matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]_{i,j}$, where $x_1, \ldots, x_n$ are local coordinates at x.*
- *A critical point $x \in X$ of f is called* **non-degenerate** *if the Hessian of f at x is an invertible matrix.*
- *f is called a* **Morse function** *if all of its critical points are non-degenerate and all its critical values are distinct.*
- *The* **Morse index** *of f at a critical point x is the number of negative eigenvalues of its Hessian at x.*

Classical Morse Theorem, Part B

Theorem (Fundamental Theorem of Morse Theory, Part B*)

If f is a Morse function, $x \in X$ is a critical point of f with critical value $v = f(x)$, and λ is the Morse index of f at x, then the change in $X_{\leq a}$ as a passes v is given, up to homotopy, by attaching a λ-cell.

Morse Data and the Full Statement

Definition

Let f be a Morse function, x a critical point, and $v = f(x)$ as before. Let ε be so small that $[v-\varepsilon, v+\varepsilon]$ contains no other critical values of f. We say that a pair of spaces (A, B) with an attaching map $h : B \to X_{\leq v-\varepsilon}$ is **Morse data** *for f at x if $X_{\leq v-\varepsilon} \cup_B A$ is homeomorphic to $X_{\leq v+\varepsilon}$.*

- We can restate the previous theorem by saying that "$(D^\lambda, \partial D^\lambda)$ is Morse data for f at x up to homotopy".
- In fact, a slightly stronger result holds: $(D^\lambda \times D^{n-\lambda}, \partial D^\lambda \times D^{n-\lambda})$ is Morse data for f at x.

Setup for Whitney Stratifications

- Not all varieties are smooth!
- Let's work with quasi-projective varieties for the sake of simplicity.
- More generally, we are interested in the situation where X is a locally closed subset of a smooth manifold M of dimension m.

Stratifications

Definition

Under these circumstances, a **stratification** *of X is a collection of locally closed subsets S_α of X such that:*

- *Each S_α is a smooth submanifold of M.*
- *$X = \bigcup_\alpha S_\alpha$ and this union is disjoint and locally finite.*
- *If S_α and S_β are distinct strata such that S_α meets $\overline{S^\beta}$, then $S_\alpha \subseteq \overline{S^\beta}$. (This is called the* **frontier condition***.)*

Example

If Q is a quasi-projective variety, then Nonsing(Q), Nonsing(Sing(Q)), Nonsing(Sing(Sing(Q))), $\ldots$ *are the strata in a stratification of Q.*

Whitney Regularity

Definition

Let $S_\alpha \subseteq \overline{S_\beta}$ be strata in a stratification and fix $x \in S_\alpha$. We say S_β is **Whitney regular** *over S_α at x if the following condition holds: Whenever we have sequences $x_n \to x$ in S_α and $y_n \to x$ in S_β such that the lines $\overline{x_n y_n}$ converge to a line ℓ in $T_x M$ and the tangent spaces $T_{y_n} S_\beta$ converge to a subspace T of $T_x M$, ℓ is contained in T.*

Non-Example

$S_\alpha = \{(0,0)\}$ *and* $S_\beta = \{(x, x \sin\left(\frac{1}{x}\right)) \mid x > 0\}$ *in* $\mathbb{R}^2$.

Whitney Stratifications

Definition

A **Whitney stratification** *is a stratification* $\{S_\alpha\}_\alpha$ *such that, for every pair of strata* $S_\alpha \subseteq \overline{S_\beta}$*,* S_β *is Whitney regular over* S_α *at every point.*

Remark

Such a stratification will also satisfy **Whitney's Condition A***: If* $x \in S_\alpha$ *and* $y_n \to x$ *is a sequence in* S_β *such that* $T_{y_n}S_\beta \to T \subseteq T_xM$*, then* $T_xS_\alpha \subseteq T$*.*

Whitney Stratifications

Definition

A **Whitney stratification** *is a stratification* $\{S_\alpha\}_\alpha$ *such that, for every pair of strata* $S_\alpha \subseteq \overline{S_\beta}$, S_β *is Whitney regular over* S_α *at every point.*

Non-Example

The singularity stratification of the **Whitney umbrella** $x^2 - zy^2 = 0$ *in* $\mathbb{C}^3$ *is not a Whitney stratification.*

However, it is true that every quasi-projective complex variety admits a Whitney stratification.

Normal Slices and Links

- Whitney stratified spaces have some nice local structure.
- Suppose we have a Whitney stratification of our locally closed subset X in M.

Definition

Let $x \in X$ and denote by S the stratum containing it. Let V be a smooth submanifold of M such that $V \cap S = \{x\}$ and V is transverse[1] *to S at x. The* **normal slice** *to S at x is $N := X \cap V \cap B$ for B a closed ball of sufficiently small radius.*

The **link** *of S at x is $L := X \cap V \cap \partial B$. Neither the normal slice nor the link depend on any of the choices made, including the choice of $x \in S$.*

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[1] I.e., T_xS and T_xV span T_xM.

Setup for Stratified Morse Theory

- Let X be a compact, Whitney-stratified subset of a smooth m-manifold M and $f : M \to \mathbb{R}$ a smooth function.
- For $a \in \mathbb{R}$, define $X_{\leq a}$ to be $X \cap f^{-1}((-\infty, a])$ and, for any stratum S, define $S_{\leq a}$ similarly.
- We now take a **critical point** of $f|_X$ to be any point $x \in X$ such that $f|_S$ has a critical point at x for S the stratum which contains x. In this case f is **non-degenerate** if it is a non-degenerate critical point of $f|_S$.

Stratified Morse Theorem, Part A

Theorem (Fundamental Theorem of Stratified Morse Theory, Part A)

If the interval $[a, b]$ contains no critical values of $f|_X$, then $X_{\leq a}$ and $X_{\leq b}$ are homeomorphic in a way that "preserves the stratification".

Conormal Vectors and Degeneracy

- Each statum S, being a submanifold, has a normal bundle given by taking the quotient of bundles $T_S M/T_S S$. Its dual is the **conormal bundle** to S - this can be identified with the subbundle of $T_S^\vee M$ consisting of cotangent vectors which vanish on $T_S S$.
- A conormal vector to S at x is called **degenerate** if, for some stratum S' whose closure contains S and sequence $y_n \to x$ in S', $T_{y_n} S'$ converges to $T \subseteq T_x M$, it vanishes on T.

Morse Functions and Morse Data, Again

Definition

$f|_X$ is called a **Morse function** *if the following hold:*

- *Its critical values are distinct.*
- *Its critical points are non-degenerate.*
- *For every critical point $x \in S$, $df_x : T_x M \to \mathbb{R}$ is a non-degenerate conormal vector to S.*

Morse Functions and Morse Data, Again

Definition

Let f be a Morse function on X, x a critical point, and $v = f(x)$. Let ε be so small that $[v - \varepsilon, v + \varepsilon]$ contains no other critical values of f. We say that a pair of "stratified" spaces (A, B) with a "stratification"-preserving attaching map $h : B \to X_{\leq v-\varepsilon}$ is **Morse data** *for f at x if $X_{\leq v-\varepsilon} \cup_B A$ is homeomorphic to $X_{\leq v+\varepsilon}$ in a way that preserves the "stratification".*

Normal and Tangential Morse Data

Definition

Let $x \in X$ be a critical point of f in the stratum S, and let N be the normal slice to S at X. Then a pair (A, B) is called **tangential Morse data** *for f at x if it is Morse data for $f|_S$ at x. On the other hand, such a pair is called* **normal Morse data** *for f at x if it is Morse data for $f|_N$ at x.*

Stratified Morse Theorem, Part B

Theorem (Fundamental Theorem of Stratified Morse Theory, Part B)

If x is a critical point of x, (A, B) is tangential Morse data for f at X, and (A', B') is normal Morse data for f at x, then $(A, B) \times (A', B') := (A \times A', B \times A' \cup A \times B')$ is Morse data for f at x.

Some Applications