(Algebraic) Worksheet 6: Poincaré Duality

2023 Geometry/Topology SEP, UW-Madison

August 4

1. (Question 4, Summer 2016) Let X be a topological space that is homotopy equivalent to a finite CW-complex. Recall that the Euler characteristic of X is defined by

$$\chi(X) = \sum_i \operatorname{Rank} H^i(X; \mathbb{Z}).$$

   Prove the following statements using only Poincaré Duality and Universal Coefficient Theorem.

   (a) $\chi(X) = \sum_i \dim H^i(X; K)$ for any field K.

   (b) Let X be a closed manifold of odd dimension. Then $\chi(X) = 0$.

   (c) Let X be a closed orientable manifold of dimension $4k + 2$ where k is an integer greater than or equal to zero. Then $\chi(X)$ is even.

2. (Question 4, Winter 2021) For $n > m > 1$, we can consider $\mathbb{CP}^m$ as a subspace of $\mathbb{CP}^n$ by

$$\mathbb{CP}^m = \{[z_0 : \cdots : z_n] \mid z_{m+1} = \cdots = z_n = 0\}.$$

   Let $M = \mathbb{CP}^n \setminus \mathbb{CP}^m$. Compute the following cohomology groups (with compact support) for $i \geq 0$,

   (a) $H^i(\mathbb{CP}^n, \mathbb{CP}^m; \mathbb{Z})$;

   (b) $H^i_c(M; \mathbb{Z})$;

   (c) $H^i(M; \mathbb{Z})$.

3. (Question 4, Winter 2019[1]) Let X be a connected, orientable, closed 4-manifold, with fundamental group $\pi_1(X) \cong \mathbb{Z}_{10}$ and Euler characteristic $\chi(X) = 4$.

   (a) Calculate $H_i(X; \mathbb{Z})$ for all integers i.

   (b) Suppose $\tilde{X}$ is a connected regular 5-sheeted covering space of X. Calculate $H_i(\tilde{X}; \mathbb{Z})$ for all integers i.

4. (Question 4, Summer 2018) Show that if M is a compact contractible n-manifold with boundary then ∂M is a homology $(n-1)$-sphere, meaning that $H_i(\partial M; \mathbb{Z}) \cong H_i(S^{n-1}; \mathbb{Z})$ for all i.

5. (Question 6, Summer 2019) Let M be an orientable n-dimensional manifold. Let $P \in M$ be any point and let $M' = M \setminus \{P\}$ be the complement.

   (a) Prove that $H^n(M', \mathbb{Q}) = 0$.

   (b) There is a natural map $f : H^i(M', \mathbb{Q}) \to H^i_c(M', \mathbb{Q})$, where $H^i_c(M', \mathbb{Q})$ is the i-th cohomology with compact support. Give an explanation of how this map can be defined.

   (c) Compute the kernel and cokernel of $f : H^i(M', \mathbb{Q}) \to H^i_c(M', \mathbb{Q})$ for every $i \in \mathbb{Z}_{\geq 0}$.

6. (Question 5, Summer 2016) Let $f : M \to N$ be a map of degree one between closed orientable n-manifolds. Prove that $f_* : H_i(M, \mathbb{Z}) \to H_i(N, \mathbb{Z})$ are splitting surjective homomorphisms. (Hint: use the naturality of cap product.)

[1] And many others — they love this one.