(Algebraic) Worksheet 9: Fiber Bundles

2023 Geometry/Topology SEP, UW-Madison

August 16

1. (Question 5, Winter 2017) Consider S^3 as a subspace of $\mathbb{C}^2$ defined by
$$S^3 = \{(x, y) \in \mathbb{C}^2; |x|^2 + |y|^2 = 1\}$$
Then there is a natural map
$$f : S^2 \to \mathbb{C}\mathbf{P}^1$$
which is the restriction map of the quotient map $\mathbb{C}^2 \setminus \{(0, 0)\} \to \mathbb{C}\mathbf{P}^1$. Show that f is not homotopy equivalent to a constant map.

2. (Question 6, Summer 2018)
   (a) Recall that a space X is *acyclic* if $\tilde{H}_i(X; \mathbb{Z}) = 0$ for all i. Prove that the suspension of an acyclic space is contractible.
   (b) Let $F \to E \to B$ be a fiber bundle with contractible fiber F. Show that the bundle projection $E \to B$ is a homotopy equivalence.

3. (Question 6, Winter 2021) Consider T^2 as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. Then multiplication by the matrix
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
induces a homeomorphism $\phi : T^2 \to T^2$. Let M be the quotient of $T^2 \times I$ by the equivalence relation generated by $(x, 0) \sim (\phi(x), 1)$.
   (a) Define an explicit CW-complex structure of M.
   (b) Compute the cohomology groups $H^\bullet(M; \mathbb{Q})$.

(c) Compute the homotopy groups $\pi_i(M)$.

4. (Question 6, Winter 2020) Let E be a fibration with base B and fiber F. Suppose that both B and F are finite CW-complexes. Prove that

$$\chi(E) = \chi(B)\chi(F)$$

where χ denotes the Euler characteristic.