(Algebraic) Worksheet 5: Cohomology

2023 Geometry/Topology SEP, UW-Madison

July 31

1. (Question 5, Summer 2019) Consider S^3 as the one point compactification of $\mathbb{R}^3$ with coordinates x, y, z. Let

   - $A \subset S^3$ be the sphere defined by $x^2 + y^2 + z^2 = 1$,
   - $B \subset S^3$ be the circle defined by $x^2 + y^2 = z = 0$.

   Let

   - X be the quotient space obtained from S^3 by collapsing A to a point,
   - Y be the quotient space obtained from S^3 by collapsing B to a point.

   Compute the cohomology groups $H^i(X, \mathbb{Z})$ and $H^i(Y, \mathbb{Z})$ for all $i \in \mathbb{Z}_{\geq 0}$.

2. (Question 5, Winter 2019)

   (a) Are the spaces $S^2 \times \mathbb{R}\mathbb{P}^4$ and $S^4 \times \mathbb{R}\mathbb{P}^2$ homotopy equivalent? Rigorously justify your answer.

   (b) Is there a retraction of $\mathbb{R}\mathbb{P}^n$ onto $\mathbb{R}\mathbb{P}^k$ for $k < n$? Rigorously justify your answer.

3. (Question 5, Summer 2018)

   (a) Show that $S^1 \times S^2$ and $S^1 \vee S^2 \vee S^3$ have isomorphic homology but are not homotopy equivalent.

   (b) Let $k < n$, and let $\mathbb{R}\mathbb{P}^k \subset \mathbb{R}\mathbb{P}^n$ be the standard inclusion. Is $\mathbb{R}\mathbb{P}^k$ and retract of $\mathbb{R}\mathbb{P}^n$? Justify your answer.

4. (Question 5, Winter 2021) Let $m, n \in \mathbb{Z}_{>0}$.

   (a) Describe the cohomology rings
   $$H^\bullet(\mathbb{RP}^m \vee \mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad H^\bullet(\mathbb{RP}^m \times \mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}).$$

   (b) Prove that $\mathbb{RP}^m \vee \mathbb{RP}^n$ can not be a retract of $\mathbb{RP}^m \times \mathbb{RP}^n$.

5. (Question 4, Summer 2020) Let X be the 3-dimension torus $T^3 = S^1 \times S^1 \times S^1$ with one point removed, and let Y be the 3-dimensional sphere S^3 with three distinct points removed. Describe the cohomology rings
   $$H^\bullet(X, \mathbb{Z}) \quad \text{and} \quad H^\bullet(Y, \mathbb{Z}).$$

6. (Question 5, Summer 2020) Let X be a connected finite CW complex, and let $f : Y \to X$ be a 2-fold covering of X.

   (a) Prove that the induced map
   $$f^*_{\mathbb{Q}} : H^\bullet(X, \mathbb{Q}) \to H^\bullet(Y, \mathbb{Q})$$
   is injective.

   (b) Is the induced map
   $$f^*_{\mathbb{Z}} : H^\bullet(X, \mathbb{Z}) \to H^\bullet(Y, \mathbb{Z})$$
   always injective? Justify your answer.

7. (Question 5, Summer 2017) Are $\mathbb{CP}^{n(n+1)/2}$ and $S^2 \times S^4 \times \cdots \times S^{2n}$ homotopy equivalent?

   Are $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ homotopy equivalent?

   (Justify all answers.)

8. (Question 4, Winter 2020) Let X be the complement of the circle
   $$\{x_1^2 + x_2^2 - 1 = x_3 = 0\}$$
   in $\mathbb{R}^3$. Describe the cohomology ring $H^*(X, \mathbb{Z})$.

9. (Question 4, Summer 2019) Let X and Y be simply connected CW complexes. Suppose $X \times \mathbb{RP}^m$ is homotopy equivalent to $Y \times \mathbb{RP}^n$. Prove that $m = n$.

10. (Question 6, Winter 2017) Let $(a_1, \dots, a_k)$ and $(b_1, \dots, b_l)$ be two sequences of positive integers with $1 \leq a_1 \leq \cdots \leq a_k$ and $1 \leq b_1 \leq \cdots \leq b_l$. Suppose there is a homotopy equivalence

$$\mathbb{C}\mathbf{P}^{a_1} \times \mathbb{C}\mathbf{P}^{a_2} \times \cdots \times \mathbb{C}\mathbf{P}^{a_k} \simeq \mathbb{C}\mathbf{P}^{b_1} \times \mathbb{C}\mathbf{P}^{b_2} \times \cdots \times \mathbb{C}\mathbf{P}^{b_l}.$$

Show that $k = l$ and $a_i = b_i$ for all i.