(Differential) Worksheet 8: de Rham Cohomology

2023 Geometry/Topology SEP, UW-Madison

August 11

1. (Question 5, Summer 2015) Let $M = \mathbb{R}^3 - (0,0,0)$. For which values of k can we assert the following?

   "Given any smooth k-form ω with $d\omega = 0$, there exists a smooth form η such that $d\eta = \omega$"?

   Explain in detail all your assertions.

2. (Question 6, Summer 2017) Let Ω be an open connected domain in $\mathbb{R}^n$.

   (a) Suppose that Ω is star-shaped with respect to a point $x_0 \in \Omega$, i.e., for every $x \in \Omega$ the line segment from x_0 to x is entirely contained in Ω. Show that any closed 1-form on Ω is exact.

   (b) Will the statement in (a) still be true if we remove the assumption that Ω is star-shaped? Please justify your answers.

3. (Question 6, Summer 2018) Let dA denote the standard area form on the standard 2-sphere S^2 (with the orientation determined by the outward-pointing normal vector), and let (x, y, z) denote the Cartesian coordinates for $\mathbb{R}^3$. Determine the values of integer $n \geq 0$ for which the form $\omega = z^n\, dA$ is exact.

4. (Question 5, Summer 2020) Consider the covering map $\pi : \mathbb{R}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

   (a) Let F be the map $\mathbb{R} \to T^2$ defined by $F(t) = \pi(t, 2t)$. Prove that the image of F is an embedded submanifold.

(The following statement may be useful, not useful, or wrong: The image of an immersion between compact manifolds is an embedded submanifold.)

(b) Let x, y be the coordinates on $\mathbb{R}^2$. Let θ be the unique 1-form on T^2 such that $\pi^*\theta = dx$. Prove that θ is closed but not exact.

(Part (a) may be useful or not useful to solve Part (b).)

5. (Question 6, Winter 2017) Let ω be an n-form on $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$ (i.e., the Euclidean n-space with its origin removed) defined by

$$\omega = |\vec{x}|^{-n-1} \sum_{j=1}^{n+1} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1},$$

where $\hat{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{\vec{0}\}$,

$$|\vec{x}| = \left(\sum_{j=1}^{n+1} x_j^2 \right)^{\frac{1}{2}},$$

and $\widehat{dx_j}$ denotes to eliminate the term dx_j. Show the following:

(a) ω is closed; that is, $d\omega = 0$.

(b) ω is *not* exact; that is, there is no $(n-1)$-form α on $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$ so that $\omega = d\alpha$.