(Differential) Worksheet 10: Lie Groups

2023 Geometry/Topology SEP, UW-Madison

August 18

1. (Question 4, Winter 2015) Prove that the special linear group

$$\mathrm{SL}(n) = \{M \in M_{n\times n}, \quad \det M = 1\}$$

is a Lie group; that is, it is a manifold and the map

$$(M, N) \to MN^{-1}$$

is smooth.

2. (Question 4, Summer 2021) Let $O(n) = \{A \in M_{n\times n}(\mathbb{R}) \mid AA^T = I_n\}$. Show that $O(n)$ is a compact manifold.

3. (Not a qual question) Compute the Lie algebras of $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, and/or $\mathrm{Sp}_n(\mathbb{R})$.

4. (Question 4, Summer 2020) Let V be an n-dimensional real vector space. Let e_i be a basis on V and let e^i be the dual basis. Define

$$\varphi = \sum_{i,j,k=1}^{n} \varphi_{ijk} e^i \otimes e^j \otimes e^k$$

as a tensor in $V^* \otimes V^* \otimes V^*$. Define G as the group

$$G = \{g \in \mathrm{GL}(n, \mathbb{R}) = \mathrm{Aut}(V) : g^*\varphi = \varphi\}.$$

What is the Lie algebra corresponding to G?