(Differential) Worksheet 9: Distributions

2023 Geometry/Topology SEP, UW-Madison

August 14

1. (Question 6, Winter 2013) Consider the 1-forms

$$\omega_1 = x^2 dx^1 - x^3 dx^4, \quad \omega_2 = x^3 dx^2 - x^2 dx^3$$

on $\mathbb{R}^4$ and the distribution $\mathcal{D}$ given by

$$\mathcal{D} = \ker \omega_1 \cap \ker \omega_2$$

on

$$U = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \mid x^2 \neq 0 \text{ or } x^3 \neq 0\}.$$

   (a) Show that $\mathcal{D}_p$ has constant rank for $p \in U$, and so defines a distribution.

   (b) Prove $\omega_i([X, Y]) = -d\omega_i(X, Y)$ for any vector fields tangent to $\mathcal{D}$ for $i = 1, 2$.

   (c) Prove that the distribution is involutive.

2. (Question 4, Winter 2021 & Summer 2013) Let f be a real-valued smooth function on $\mathbb{R}^3$ and consider the vector fields defined by

$$X = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}, \quad Y = \frac{\partial f}{\partial z}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial z}, \quad Z = \frac{\partial f}{\partial z}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial z}.$$

   Define a subspace $W(p) = \{X(p), Y(p), Z(p)\}$ for $p \in \mathbb{R}^3$.

   (a) Show that $W(p)$ has dimension 2 if p is a regular point of f, and 0 otherwise.

   (b) Let N be the set of regular points of f. Show that W defines an integrable distribution on N.

(c) Show that f is constant on any connected integral manifold of the distribution W on N.

3. (Question 6, Summer 2015) Let $V = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ be a nowhere zero C^∞ vector field in $\mathbb{R}^3$. Show that the following three statements are equivalent.

   (a) The distribution orthogonal to V is integrable in some neighborhood of the origin.

   (b) There exists a nowhere zero C^∞ function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathrm{curl}(fV) = 0$ on some neighborhood of the origin.

   (c) $V \cdot \mathrm{curl}(V) \equiv 0$ in some neighborhood of the origin.

4. (Question 5, Winter 2015) Let $\Delta_1, \ldots, \Delta_k$ be a family of involutive distributions in M with dimensions $d_1, \ldots, d_k$ respectively. Assume that for every $p \in M$

$$T_pM = (\Delta_1)_p \oplus \cdots \oplus (\Delta_k)_p$$

and that $\Delta_i \oplus \Delta_j$ is also involutive for every i, j. Show that around each $p \in M$ there are coordinates (x, U) such that Δ_1 is generated by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{d_1}}$, Δ_2 is generated by $\frac{\partial}{\partial x_{d_1+1}}, \ldots, \frac{\partial}{\partial x_{d_1+d_2}}$, and so on.