Worksheet 1: Fundamental Groups

2023 Geometry/Topology SEP, UW-Madison

July 17

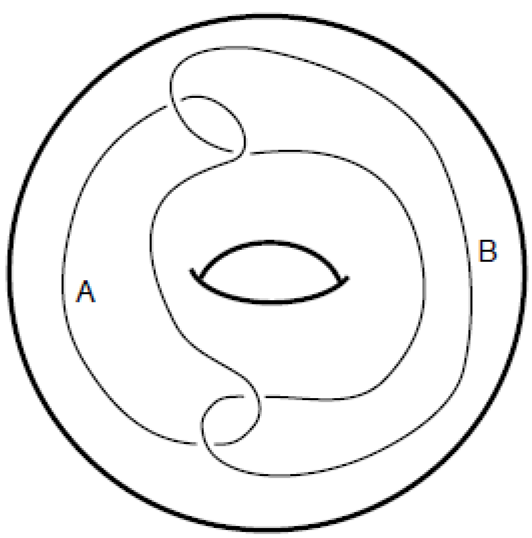
1. (Part of Question 2, Summer 2019) Let W be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^{10}$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$.

   Compute the fundamental group of W.

2. (Part of Question 1, Winter 2020) Consider X the "bagel with cream cheese" 2-complex: slice a bagel to obtain two halves, apply cream cheese and consider the 2-complex you get when you stick the two halves together and count the cream cheese layer. (This is just a torus with an annulus glued into it).

   Compute the fundamental group of X.

3. (Question 2, Summer 2017) Consider the two curves A and B in the solid torus $T \cong D^2 \times S^1$ in $\mathbb{R}^3$ as shown in the figure. Draw a picture that shows that A bounds a disk in $\mathbb{R}^3 - B$. Show that A is not null-homotopic in $T - B$.



4. (Part of Question 2, Summer 2020) Let X be the 2-torus and Y be a compact surface of genus 2. Let $\alpha \subset X$ be a circle such that $X \setminus \alpha$ is connected, and let $\beta \subset Y$ be a

circle such that $Y \setminus \beta$ is disconnected.[1] Moreover, assume that both α and β have open neighborhoods in X and Y respectively that are homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Let W be the topological space formed by gluing X and Y via a homeomorphism $f : \alpha \to \beta$. Compute $\pi_1(W)$.[2]

5. (Part of Question 3, Summer 2017) Let X be a closed genus-2 surface. Let α be a nonseparating circle in X (*i.e.* $X \setminus \alpha$ is connected), and let β be a separating circle in X (*i.e.* $X \setminus \beta$ is disconnected) that is disjoint from α and that is not null-homotopic. Let Y be the space obtained from X by identifying α and β via a homeomorphism $\alpha \cong \beta$ (there is a choice of homeomorphism here but don't worry about it, just pick one). Compute the fundamental group of Y.

6. (Part of Question 2, Winter 2019) Let X be the quotient space of an annulus obtained by identifying antipodal points on the outer circle and identifying points on the inner circle which are $2\pi/3$ radians apart. Find $\pi_1(X)$.

7. (Question 1, Winter 2022) For relatively prime positive integers m and n, the *torus knot* $K_{m,n} \subset \mathbb{R}^3$ is the image of the embedding $f : S^1 \to S^1 \times S^1 \subset \mathbb{R}^3$, $f(z) = (z^m, z^n)$, where the torus $S^1 \times S^1$ is embedded in $\mathbb{R}^3$ in the standard way. (Here we are thinking of S^1 as the unit circle in $\mathbb{C}$ and z^m and z^n are polynomials of a complex variable.) Compute $\pi_1(\mathbb{R}^3 \setminus K_{m,n})$.

[1] Something to note here: I'm keeping the language as faithful to the exam is possible because I think it's useful to be prepared for what it actually looks like, but as written this question *is* under-specified! The author was probably thinking of a homotopically essential (i.e., not null-homotopic) circle β, as in the next problem, but didn't actually write that part down. If you encounter an ambiguity like this in the actual exam, you can ask the proctor or, if you feel the intent is obvious, just state the added assumption clearly in your solution. (Probably best to check, though!)

[2] It's not immediately clear here, and in the next question, why the answer doesn't depend on the choice of α and β. For now, it's fine to go ahead and assume this — we'll talk more about how to show it later once we've reviewed some extra machinery. On an exam, you *might* be able to get away with assuming it, but it's probably best to be safe.