(Differential) Worksheet 5: Manifolds

2023 Geometry/Topology SEP, UW-Madison

July 31

1. (Question 6, Summer 2020) Prove that any smooth map from a 2-dimensional manifold M to the standard sphere S^3 is homotopic to a constant map.

   (The following statements may be useful, not useful, or wrong: 1. Any continuous map from M to $\mathbb{R}^3$ is homotopic to a constant map. 2. There is a theorem which holds for smooth maps but not for continuous maps.)

2. (Question 5, Winter 2018) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\Phi(x, y) = x^2 - y^2$.

   (a) Show that $\Phi^{-1}(0)$ is **not** an embedded submanifold of $\mathbb{R}^2$.

   (b) Can $\Phi^{-1}(0)$ be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$?

3. (Question 5, Summer 2017) Let $\mathbb{S}^n$ be the unit n-sphere.

   (a) Prove that the tangent bundle $T\mathbb{S}^3$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{R}^3$.

   (b) For any n, is the tangent bundle $T\mathbb{S}^n$ always diffeomorphic to $\mathbb{S}^n \times \mathbb{R}^n$? Please justify your answers.

4. (Question 5, Winter 2017) Recall that $2n \times 2n$ real matrices can be identified with the points of $\mathbb{R}^{4n^2}$ in an obvious manner. Thus the set of all nonsingular $2n \times 2n$ real matrices, denoted by $\mathrm{GL}(2n, \mathbb{R})$, has a natural manifold structure as the open subset of $\mathbb{R}^{4n^2}$ where the determinant function does not vanish.

   Consider the set

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{ A \in \mathrm{GL}(2n, \mathbb{R}) \colon A^T \Omega A = \Omega \right\},$$

where A^T is the transpose of the matrix A and

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

with I_n the $n \times n$ identity matrix. Prove that $\mathrm{Sp}(2n, \mathbb{R})$ has a manifold structure in which the inclusion map $i : \mathrm{Sp}(2n, \mathbb{R}) \hookrightarrow \mathrm{GL}(2n, \mathbb{R})$ is a smooth imbedding.

5. (Question 4, Winter 2018) Let M be a non-empty topological n-manifold without boundary with $n \geq 1$. If M has a smooth structure, show that M has uncountably many smooth structures.

6. (Question 4, Summer 2017) The n-dimensional real projective space, denoted by $\mathbb{RP}^n$, is defined as the set of 1-dimensional linear subspaces of the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$. We give it the quotient topology given by the map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ sending $x \in \mathbb{R}^{n+1} \setminus \{0\}$ to the line through x and 0. Show that $\mathbb{RP}^n$ has a smooth manifold structure.