Worksheet 2: Covering Spaces

2023 Geometry/Topology SEP, UW-Madison

July 19

1. (Question 1, Summer 2020) Let $\mathbb{Z}_{10}$ act on $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ by $(z, w) \mapsto (\mu z, \mu w)$, where μ is a tenth root of unity. Denote by L the quotient space $\mathbb{S}^3 / \mathbb{Z}_{10}$.
   (a) Compute the fundamental group of L.
   (b) Describe all covering spaces of L.
   (c) Show that any map $L \to \mathbb{S}^1$ is null-homotopic.
2. (Question 3, Winter 2018) Let Y be the wedge of two circles. For each of the following groups G, either find a connected four-fold covering space $f : \tilde{Y} \to Y$ such that the group of deck transformations of $\tilde{Y}$ is G or prove that such a cover does not exist.
   (a) $G = S_4$
   (b) $G = \mathbb{Z}/4\,\mathbb{Z}$
   (c) $G = \mathbb{Z}/3\,\mathbb{Z}$
   (d) $G = \mathbb{Z}/2\,\mathbb{Z}$
   (e) G is trivial
3. (Question 3, Winter 2017) Let X be a path connected cell complex and suppose that there is a map $X \to S^1$ that is not null-homotopic. Let x be a zero-cell of X.
   (a) Prove that $X \bigvee_x X$ has an irregular cover of degree three.
   (b) Does $X \bigvee_x X$ admit a retract onto a wedge of two circles? Prove your answer.
4. (Question 2, Winter 2020) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation
   $$\phi(x, y) = (2x, y/2).$$
   This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 - \{0\}$.
   (a) Show that the action is a covering space action.
   (b) Compute $\pi_1(X/\mathbb{Z})$.
   (c) Show that the orbit space $X/\mathbb{Z}$ is not Hausdorff. (**Hint:** the quotient space is a union of four subsets homeomorphic to $S^1 \times \mathbb{R}$ coming from the complementary components of the x-axis and the y-axis.)
5. (Question 1, Summer 2019) Give examples (and justify your answer) of a covering space Y of a Klein bottle K where:
   (a) Y is a torus and the cover is normal
   (b) Y is a torus and the cover is not a normal cover
   (c) Y is a Klein bottle and the cover is not a normal cover