(Algebraic) Worksheet 7: Homotopy Groups

2023 Geometry/Topology SEP, UW-Madison

August 9

1. (Question 6, Winter 2018)

   (a) Let T be the torus $S^1 \times S^1$. Find two homeomorphic subsets A and B of T such that $\pi_2(T, A)$ and $\pi_2(T, B)$ are not isomorphic.

   (b) Let G be the dihedral group of order 12. Explicitly construct a space X and a subspace $A \subset X$ such that $\pi_2(X, A) \cong G$.

2. (Question 5, Winter 2018) Show that a simply-connected 3-manifold is homotopy-equivalent to the 3-sphere.

3. (Question 5, Winter 2020) Let $f : S^m \times S^n \to S^{m+n}$ be the quotient map collapsing $S^m \vee S^n$ to a point. Prove than for any $m, n > 0$, the map f induces the zero map on all homotopy groups, but f is not nullhomotopic.

4. (Question 5, Summer 2021)

   (a) Show that a map $\varphi : \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ that induces the identity map on $H^2(\mathbb{CP}^\infty; \mathbb{Z})$ is homotopic to the identity.

   (b) Let X be a cell complex with a subcomplex $Y \cong \mathbb{CP}^\infty$. Show that if the inclusion $\iota : Y \to X$ is such that $\iota^* : H^2(X; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$ is an isomorphism, then Y is a retract of X.

5. (Question 4, Winter 2017) Prove or disprove the following statement.

   If X is a CW-complex with finitely many cells, then $\pi_2(X)$ is a finitely-generated abelian group.

6. (Question 6, Summer 2020) Let X be a finite CW-complex. Suppose that X is n-connected, that is, X is connected and $\pi_i(X) = 0$ for all $1 \leq i \leq n$. Prove that, for any $1 < k < n$, the k-skeleton X^k of X is homotopy equivalent to a bouquet of k-spheres.

7. (Question 6, Winter 2019) Suppose that the cell complex X contains a subcomplex S^1 (the circle) such that the inclusion $S^1 \hookrightarrow X$ induces an injection $H_1(S^1;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ with image a direct summand of $H_1(X;\mathbb{Z})$. Show that S^1 is a retract of X.

8. (Question 4, Summer 2017)

   (a) Let M be a compact orientable manifold with boundary. Is there a retraction of M onto its boundary ∂M? Explain your answer.

   (b) Let M be the surface of genus 2. Compute the homotopy groups of $M \vee S^1$.

   (c) Give an example of two connected surfaces M and N and two non-homotopic maps $f, g : M \to N$ such that, for all n, it is the case that $f_n = g_n : \pi_n(M) \to \pi_n(N)$, where f_n and g_n are the maps induced by f and g, respectively.