Worksheet 3

2023 Geometry/Topology SEP, UW-Madison

July 24

1. (The rest of Question 2, Winter 2019) Let X be the quotient space of an annulus obtained by identifying antipodal points on the outer circle and identifying points on the inner circle which are $2\pi/3$ radians apart. Find $H_i(X;\mathbb{Z})$ for all i.

2. (Question 1, Winter 2018) Let X be the quotient space of the Klein bottle obtained by identifying two distinct points. Compute the fundamental group and all homology groups of X.

3. (Question 2, Summer 2021) Let $S \subset \mathbb{R}^3$ be the surface obtained by rotating the circle $(x-2)^2 + y^2 = 1$ about the y-axis. Let X be the quotient space of $\mathbb{R}^3$ with S collapsed to a point. Compute $H_i(X,\mathbb{Z})$ for all nonnegative integers i.

4. (Question 3, Winter 2019) Compute the homology groups of $S^{4n} \times S^{2n}$ for all $n \geq 0$.

5. (Question 2, Winter 2017)[1]

   (a) Let X and Y be closed surfaces of genus 2. Let α be a homotopically essential separating circle in X. Let β be a nonseparating circle in Y. (Assume that α and β have neighborhoods homeomorphic to $S^1 \times \mathbb{R}$.) Let Z be the space obtained by gluing X and Y together by identifying α and β by a homeomorphism.

   Compute the integral homology groups of Z.

[1] For the next three questions, as on Worksheet 1, you need not justify your choice of α and β for now. We will be ready to discuss the necessary argument on Wednesday.

(b) Let F be a closed surface embedded in S^3 with a neighborhood homeomorphic to $F \times \mathbb{R}$. Let W be the space obtained by gluing two copies of S^3 together along their copies of F using some homeomorphism $F \to F$.

Compute the integral homology groups of W.

6. (The rest of Question 2, Summer 2020) Let X be the 2-torus and Y be a compact surface of genus 2. Let $\alpha \subset X$ be a circle such that $X \setminus \alpha$ is connected, and let $\beta \subset Y$ be a circle such that $Y \setminus \beta$ is disconnected. Moreover, assume that both α and β have open neighborhoods in X and Y respectively that are homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Let W be the topological space formed by gluing X and Y via a homeomorphism $f : \alpha \to \beta$. Compute $H_*(W; \mathbb{Z})$.

7. (The rest of Question 3, Summer 2017) Let X be a closed genus-2 surface. Let α be a nonseparating circle in X (*i.e.* $X \setminus \alpha$ is connected), and let β be a separating circle in X (*i.e.* $X \setminus \beta$ is disconnected) that is disjoint from α and that is not null-homotopic. Let Y be the space obtained from X by identifying α and β via a homeomorphism $\alpha \cong \beta$ (there is a choice of homeomorphism here but don't worry about it, just pick one). Compute the homology groups of Y.

8. (Question 3, Summer 2019) Let X be a totally disconnected topological space (e.g. a cantor set) and let A be an arbitrary subset of X. Determine all groups and homomorphisms in the long exact homology sequence of (X, A).

9. (Question 3, Winter 2022) Let G be a finitely generated abelian group, and fix $n \geq 1$. Construct a CW-complex X such that $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for all $i \neq n$. More generally, given finitely generated abelian groups $G_1, G_2, \cdots, G_k$, construct a CW-complex X whose homology groups are $H_i(X) = G_i$, $i = 1, \cdots, k$, and $\tilde{H}_i(X) = 0$ for all $i \notin \{1, 2, \cdots, k\}$.