(Diffential) Worksheet 7: Lie Derivatives, Orientation, and Integration

2023 Geometry/Topology SEP, UW-Madison

August 7

1. (Question 4, Summer 2015) Show that if M is a manifold, TM is always orientable.

2. (Question 6, Summer 2021) Let M and N be m and n dimensional oriented manifolds and with $p \in M$ and $q \in N$.

   (a) Show that $T_{(p,q)}(M \times N) \simeq T_pM \times T_qN$

   (b) Show that $M \times N$ can be given an orientation by declaring that $v_1, \ldots, v_m, w_1, \ldots w_n$ in $T_{(p,q)}(M \times N)$ is positively oriented if both $v_1, \ldots, v_m$ and $w_1, \ldots w_n$ are positively oriented in T_pM and T_qN respectively.

   (c) Show that if ω and η are m and n forms with compact support on M and N respectively then

   $$\int_{M \times N} \pi_M^* \omega \wedge \pi_N^* \eta = \int_M \omega \int_N \eta$$

   where $\pi_M : M \times N \to M$ is projection and similarly for π_N.

3. (Question 4, Summer 2019) Prove *Cartan's Magic Formula* for a **one form** α on a manifold M:

   $$\mathcal{L}_X \alpha = d\iota_X(\alpha) + \iota_X d(\alpha) \ .$$

   X is a (smooth) vector field, $\mathcal{L}_X \alpha$ is the Lie derivative, d is the exterior derivative, and ι_X denotes the interior product (contraction) with X.

4. (Question 6, Winter 2021 — part (b) was also Question 6, Winter 2018) Let M be a closed n-dimensional smooth manifold (without boundary). Let V be a smooth vector field.

   (a) Prove Cartan's magic formula: for any smooth differential form α on M,
   $$\mathcal{L}_V\alpha = V \lrcorner\, (d\alpha) + d(V \lrcorner\, \alpha),$$
   where $\mathcal{L}_V\alpha$ means taking the Lie derivative of α with respect to V.

   (b) Let ω be a smooth n-form on M. Prove that
   $$\int_M \mathcal{L}_V\omega = 0.$$

5. (Question 5, Summer 2019) Consider the map
   $$\omega : S^2 \to \bigwedge^2 T^*S^2 \qquad \omega_p(U,V) = p \cdot (U \times V)\ .$$
   Where $p \in S^2$ and $U, V \in T_pS^2$. $\times$ denotes the usual cross product on $\mathbb{R}^3$.

   (a) Prove that ω is a smooth 2 form on the sphere.

   (b) Evaluate
   $$\int_{S^2} \omega\ .$$

6. (Question 4, Winter 2016) Let X, Y be two commutative smooth vector fields on a smooth manifold M without boundary.

   (a). Show that both X and Y are complete if M is compact. In other words, there exist global flows generated by X and Y.

   (b). Let θ and φ be global flows generated by X and Y. Show that $\theta_t \circ \varphi_s = \varphi_s \circ \theta_t$ for each $s, t \in \mathbb{R}$.

   (c). On a non-compact manifold, find a smooth vector field which is not complete .

7. (Question 5, Summer 2021)

   (a) On $\mathbb{R}^3$ define vector fields $X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}$. Compute $[X,Y]$.

(b) Let $\mathcal{D} \subseteq \mathcal{T}(M)$ be a subset of the set of all smooth vector fields on a smooth manifold M and for $q \in M$ let

$$\mathcal{D}_q := \{X_q \in T_qM \mid X \in \mathcal{D}\}.$$

We say an embedded submanifold $\phi : N \to M$ is special if for each $p \in N$ the image of its tangent space $\phi_*(T_pN)$ equals $\mathcal{D}_{\phi(p)}$. Suppose that for each $q \in M$ there is a special embedded submanifold containing it. Show that that $\mathcal{D}$ satisfies the following property

$$X, Y \in \mathcal{D} \Rightarrow [X, Y] \in \mathcal{D}.$$

8. (Question 5, Winter 2022) Suppose V, W are smooth vector fields on M. We say that V is *invariant* under the flow (call it θ) of W if V is θ_t related to itself for all t (i.e. V is related to V under that map θ_t for all t).

(a) Show that $[V, W] = 0$ iff V is invariant under the flow of W.

(b) Show that if $V = \frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z}$ and $W = \frac{\partial}{\partial y} - \frac{x}{x^2+y^2}\frac{\partial}{\partial z}$ are vector field on $M = \mathbb{R}^3 \setminus \{(0,0,z) \mid z \in \mathbb{R}\}$ then $[V, W] = 0$ but if θ, ψ are the associated flows then there is $p \in M$ and $s, t \in \mathbb{R}$ such that $\theta_t \circ \psi_s(p) \neq \psi_s \circ \theta_t(p)$. (i.e. the naive definition of flows commuting doesn't work)