Worksheet 4: More Homology

2023 Geometry/Topology SEP, UW-Madison

July 26

1. (Question 3, Summer 2018) A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ is called an *even map*. Show that an even map has even degree, and this degree is in fact zero when n is even. When n is odd, show there exist even maps of any given even degree.

2. (Question 1, Winter 2017)

   (a) Is there a continuous map $f : \mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1}$ with no fixed points? Explain. (SEP add-on: What about $\mathbb{RP}^{2k} \to \mathbb{RP}^{2k}$?)

   (b) Construct a homotopically essential map $S^1 \times S^1 \to S^2$ and justify your construction.

3. (Question 1, Summer 2018) For a given sequence of continuous maps

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

   define the mapping telescope as the quotient space

$$M := \left(\bigsqcup_{i \geq 1} X_i \times [0,1] \right) / \big((x_i, 1) \sim (f_i(x_i), 0) \big)$$

   obtained from the disjoint union of the cylinders $X_i \times [0,1]$ via the identification of $(x_i, 1) \in X_i \times \{1\}$ with $(f_i(x_i), 0) \in X_{i+1} \times \{0\}$. Compute the homology groups of M in the case when each X_i is the n-sphere S^n (for $n \geq 1$ fixed) and each $f_i : S^n \to S^n$ is a map of degree i ($i \geq 1$).

4. (Question 3, Summer 2020) The transfer homomorphism τ of a double covering $p : \tilde{X} \to X$ assigns to each singular chain in X the sum of its two lifts in $\tilde{X}$.

(a) Show that the sequence

$$0 \to C_*(X;\mathbb{Z}_2) \to C_*(\tilde{X};\mathbb{Z}_2) \to C_*(X;\mathbb{Z}_2) \to 0$$

is an exact sequence of chain complexes. (Where the first map is τ and the second map is induced by p.)

(b) Use the corresponding long exact sequence in homology to prove that an odd map from $\mathbb{S}^n$ to itself has odd degree. (A map $f : \mathbb{S}^n \to \mathbb{S}^n$ is *odd* iff $f(-x) = -f(x)$.)

5. (Question 1, Winter 2019)

   (a) Let S_g be the closed orientable surface of genus g. Show that if a and b are positive, and $f : S_b \to S_a$ is a map of non-zero degree such that $f_* : \pi(S_b) \to \pi_1(S_a)$ is injective, then $b \geq a$.

   (b) Find an irregular (*i.e.* non-normal) cover of the Klein bottle.

6. (Question 2, Winter 2021) Compute the homology groups of the following spaces X and Y (with coefficients in $\mathbb{Z}$).

   (a) Start with three n-dimensional spheres, A, B, and C. Let X be the quotient of $A \sqcup B \sqcup C$ defined by pinching the north pole of A with the south pole of B; the north pole of B and the south pole of C; the north pole of C and the south pole of A.

   (b) Consider S^3 as the union sphere in $\mathbb{R}^4$. Let Y be the intersection of S^3 and the union of all coordinate hyperplanes. In other words,

   $$Y = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2+x_2^2+x_3^2+x_4^2 = 1 \text{ and } x_1x_2x_3x_4 = 0\}.$$

7. (Question 1, Winter 2021) Let X be the wedge sum of the real projective space $\mathbb{R}\,\mathbb{P}^2$ and the circle S^1. Describe all connected double covers of X, and compute their homology groups (with coefficients in $\mathbb{Z}$).

8. (The rest of Question 2, Summer 2019) Let W be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^{10}$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$.

   (a) Recall from Worksheet 1 that the fundamental group of W is trivial.

(b) Compute the homology of W with coefficients in $\mathbb{Z}$.

(c) Is W homeomorphic to the 2-sphere S^2? Justify your answer.

9. (Question 2, Winter 2018) Determine which of the following spaces are homotopy equivalent to each other.

   (a) $S^1 \vee S^1$

   (b) The complement in S^3 of the Hopf Link

   (c) The complement in S^3 of the unlink with two components

   (d) The complement in $\mathbb{R}^3$ of two parallel lines

   (e) The complement in $\mathbb{R}^3$ of two intersecting lines

10. (The rest of Question 1, Winter 2020) Consider X the "bagel with cream cheese" 2-complex: slice a bagel to obtain two halves, apply cream cheese and consider the 2-complex you get when you stick the two halves together and count the cream cheese layer. (This is just a torus with an annulus glued into it).

    Is this space homotopy equivalent to the 3-Torus? Explain.

11. (Question 3, Summer 2021) Prove or disprove the following statement:

    (a) All homology groups $H_i(X, \mathbb{Z})$ of a connected compact topological space X are finitely generated.

    (b) For any connected compact CW-complex C, its homology groups $H_i(X, \mathbb{Z})$ are all finitely generated.

    (c) Let Y be a closed subspace of X. If all the homology groups $H_i(X, \mathbb{Z})$ and $H_i(Y, \mathbb{Z})$ are finitely generated, then so are the groups $H_i(X \setminus Y, \mathbb{Z})$.