(Differential) Worksheet 6: Flows, Forms, and More

2023 Geometry/Topology SEP, UW-Madison

August 2

1. (Question 4, Summer 2018)

   (a) Show that the subset M of $\mathbb{R}^3$ defined by the equation

   $$(1 - z^2)(x^2 + y^2) = 1$$

   is a smooth submanifold of $\mathbb{R}^3$.

   (b) Define a vector field on $\mathbb{R}^3$ by

   $$V = z^2 x \frac{\partial}{\partial x} + z^2 y \frac{\partial}{\partial y} + z(1 - z^2)\frac{\partial}{\partial z}.$$

   Show that the restriction of V to M is a tangent vector field to M.

   (c) Show that the family of maps

   $$\phi_t(x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z)$$

   restricts to a one-parameter family of diffeomorphisms of M. For each t, determine the vector field $d\phi_t(V)$ on M.

2. (Question 4, Summer 2016) Let f, g be smooth maps from S^{2n} to S^{2n}, and let $h = f \circ g$. Show that one of the smooth maps among f, g and h must have a fixed point.

3. (Question 5, Summer 2018) Consider the submanifold $\iota : M \hookrightarrow \mathbb{R}^3$ given by $x^2 + y^2 - z^2 = 1$.

(a) Show that the 2-form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ restricts to an area form on M, i.e., a 2-form which never vanishes.

(b) Let X be the vector field on $\mathbb{R}^3$ given by

$$X = \frac{xz}{1+z^2}\frac{\partial}{\partial x} + \frac{yz}{1+z^2}\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

and let Y be the restriction of X to M (Note that Y is tangent to M, although you are allowed to take this fact for granted). Does the flow of Y on M preserve $\iota^*(\omega)$? Please justify your answer.

4. (Question 4, Winter 2017) Show that there exists a smooth vector field on S^n that vanishes at exactly one point. Here

$$S^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1 \right\}$$

is given the standard manifold structure.

5. (Question 5, Winter 2021) Let M be a closed n-dimensional smooth manifold, and $\alpha_1, \ldots, \alpha_k$ with $k \leq n$ be one-forms on M that are point-wise linearly independent.

(a) Let $p \in M$ be given. Construct a coordinate $(x_1, \ldots, x_n)$ at p such that

$$dx_1(p) = \alpha_1(p), \ldots, dx_k(p) = \alpha_k(p).$$

(b) Using the last step, prove that one-forms $\beta_1, \ldots, \beta_k$ on M satisfy the equation

$$\sum_{i=1}^{k} \alpha_1 \wedge \beta_i = 0$$

if and only if there exist smooth functions f_{ij} on M such that

$$f_{ij} = f_{ji}, \quad \beta_i = \sum_{j=1}^{k} f_{ij}\alpha_i.$$

6. (Question 5, Summer 2016) A symplectic structure on a $2n$-dimensional smooth closed manifold M^{2n} is a closed, non-degenerate 2-form ω, i.e., $d\omega = 0$ and $\omega^n \neq 0$ everywhere.

(a) Construct an explicit symplectic structure on $S^2 \times S^2$.

(b) Construct an explicit symplectic structure on T^*S^2, the cotangent bundle of S^2.

(c) Does $S^3 \times S^1$ have a symplectic structure? Justify your assertion.

7. (Question 6, Winter 2016) A smooth 2-form ω on a manifold M is called symplectic if ω is closed and nondegenerate, i.e., $d\omega = 0$ and the map

$$\omega_p : \quad T_pM \mapsto T_p^*M, \quad X \mapsto \omega(X, \cdot) \tag{1}$$

is an isomorphism for every point $p \in M$.

(a). (M, ω) is called a symplectic manifold if M is smooth and ω is symplectic. Let (M^n, ω) be a symplectic manifold. Show that n is even.

(b). Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds of the same dimension $n = 2k$. Show that $M_1 \times M_2$ admits a natural symplectic form $\omega = \pi_1^*(\omega_1) + \pi_2^*(\omega_2)$, where π_1 and π_2 are the projection to the M_1 and M_2 respectively.

(c). Suppose f is a diffeomorphism from M_1 to M_2, $N = \{(x, f(x))|x \in M_1\}$. Show that $\omega|_N = 0$ if and only if $f^*(\omega_2) = \omega_1$.