Alex Hof
MATH 863
Spring 2020
Final Paper: Tropicalization via Blueprints

The aim of this note is to give a brief overview of the rudiments of Oliver Lorscheid's theory of blueprints and blue schemes, the exposition largely being based upon his lecture notes [1] and preprint [2].

Let me begin by laying out the background as I understand it. Broadly speaking, the central idea of tropical geometry is to take algebro-geometric objects defined over some field $k$ and study simpler, skeletonized simulacra of them, which are constructed using some fixed valuation (or, equivalently, non-Archimedean absolute value) on $k$. In practice, this can take a number of different forms: there are the notions of the tropicalization of an affine variety as, alternately, the image of the closed points under the valuation or the joint tropical vanishing locus of the polynomials in the corresponding ideal; the Kajiwara-Payne tropicalization extending this notion to subvarieties of toric varieties; the Giansiracusa tropicalization endowing this last notion with something of a scheme structure; the Berkovich analytification we have seen in class; and various and sundry others with which I am less familiar. The aim of Lorscheid's theory is to give a common framework within which all of these disparate notions can be understood.

One candidate which initially seems appealing is the **category of ordered semirings with subadditive homomorphisms** — that is, the category where the objects are semirings with fixed partial orders respecting the operations and their units and the morphisms are order-preserving multiplicative maps $f$ such that $f(a+b) \leq f(a) + f(b)$. In this context, a valuation on a field $k$ is simply a morphism $v : k \to \mathbb{T}$, where $\mathbb{T}$ is the tropical semiring (with the max-plus convention and the obvious ordering) and $k$ is given the trivial partial order. It is then natural to wonder whether tropicalization can be understood as a pullback along the corresponding map $\operatorname{Spec} \mathbb{T} \to \operatorname{Spec} k$ (for some appropriate notion of "Spec"). However, there are apparently difficulties which prevent the construction of this kind of a "tensor product of ordered semirings with subadditive homomorphisms".

To surmount this issue, Lorscheid considers a slightly more flexible setting, the **category of ordered blueprints**, where the tensor product makes sense and has the properties we want. The most straightforward definition is as follows. For the objects, we have pairs $(B^\bullet, B^+)$ where $B^+$ is an ordered semiring and $B^\bullet$ is a multiplicatively closed subset of $B^+$ which contains 0 and 1. For the morphisms $(B^\bullet, B^+) \to (C^\bullet, C^+)$, we have morphisms $B^+ \to C^+$ of ordered semirings — N.B., these are now true semiring morphisms which preserve the order, not merely the subadditive morphisms discussed above — which map $B^\bullet$ into $C^\bullet$.

For any ordered blueprint $(B^\bullet, B^+)$, we can think of the set $B^\bullet$, loosely, as "the collection of elements of $B^+$ we actually care about". Note that $B^\bullet$ has the structure of a commutative **monoid-with-zero** — that is, a commutative monoid with an element 0 such that $0 \cdot m = m \cdot 0 = 0$ for all $m$. To emphasize the primacy of this subset, Lorscheid introduces a second category, that of **axiomatic ordered blueprints**, which is equivalent to the first and, although less intuitive to define, ends up being easier to work with in many cases.

An axiomatic ordered blueprint is a monoid-with-zero $A$ together with a preorder on the set of finite formal sums of elements of $A$ which respects the multiplication in $A$ and the formal addition, restricts to a partial order on $A$, and makes $0 \in A$ both precede and follow the empty sum (so that, if we mod out by the induced equivalence relation to make the preorder into a partial order, the two are identified). A morphism of axiomatic ordered blueprints is a morphism of the underlying monoids-with-zero (i.e., a map of sets preserving multiplication, the zero element, and the unit) so that the induced map on sets of finite formal sums preserves the preorder.

To see the equivalence with our first definition of an ordered blueprint, start with $(A, \leq)$ and mod the set of finite formal sums out by the equivalence relation induced by the preorder. This yields an ordered semiring $B^+$ generated as a semiring by the underlying monoid-with-zero $A$, which we take to be our $B^\bullet$. It is straightforward to check that this construction is functorial and that an inverse functor can be defined. Note also that, whichever formulation of the category we use, there are natural ways to embed in it the categories of semirings, ordered semirings more generally, monoids-with-zero, and ordered monoids-with-zero more generally. Finally, although we will not go into detail here, there is a notion of "the spectrum of an ordered blueprint", and as in the scheme case we can glue together along localizations to get the **category of ordered blue schemes**.

Having established our setting, we must now retrieve the notion of a valuation, since we have insisted on additivity and not merely subadditivity in our morphisms. To do so, we introduce for an axiomatic ordered blueprint $B = (A, \leq)$ its **monomialization** $B^{\mathrm{mon}} := (A, \leq')$, where $\leq'$ is the equivalence relation defined by "throwing away all the relations $\sum_i a_i \leq \sum_i b_i$ except those of the form $a \leq \sum_i b_i$ for $a, b_i \in A$" in some suitably precise sense. Note that the notion of monomialization can be extended to ordered blue schemes by gluing together local monomializations.

Then, for $k$ a field, a map $v : k \to \mathbb{T}$ of monoids-with-zero is a valuation exactly when the composition $k^{\mathrm{mon}} \to k \xrightarrow{v} \mathbb{T}$ is a morphism of ordered semirings, where $k$ is identified with the ordered blueprint $(k, k)$ with trivial order and $\mathbb{T}$ with the ordered blueprint $(\mathbb{T}, \mathbb{T})$ with the usual order on $\mathbb{T}$. For any ordered blue $k$-scheme $X$, we can now define the **tropicalization** of $X$ as the ordered blue $\mathbb{T}$-scheme $\mathrm{Trop}_v(X) := X^{\mathrm{mon}} \times_{\mathrm{Spec}\, k^{\mathrm{mon}}} \mathrm{Spec}\, \mathbb{T}$, where $\times_{\mathrm{Spec}\, k^{\mathrm{mon}}}$ denotes the 'fiber product' which arises by locally taking tensor products of ordered blueprints.

It remains to demonstrate that this notion is in some sense useful by tying it back to the notions of tropicalization which actually appear in practice. To begin, let $X$ be an affine $k$-scheme. Then it is not difficult to show that the set $\mathrm{Trop}_v(X)(\mathbb{T})$ of $\mathbb{T}$-valued points — i.e., maps $\mathrm{Trop}_v(X) \to \mathrm{Spec}\, \mathbb{T}$ — of the tropicalization is naturally identified with the set of points of the Berkovich analytification $X^{\mathrm{an}}$ along the non-Archimedean absolute value corresponding to $v$. Moreover, Lorscheid has a general definition of a so-called **fine topology** on the $\mathbb{T}$-valued points of an ordered blue scheme which in this case retrieves the topology of the Berkovich analytification. The same remarks almost apply as well for arbitrary $k$-schemes, but due to some unpleasantness with the gluing one must commit to a particular **blue model** for $X$, and different choices may yield different tropicalizations. However, the space of $\mathbb{T}$-valued points for any of these tropicalizations remains homeomorphic to $X^{\mathrm{an}}$.

There is, likewise, a realization of the Kajiwara-Payne tropicalization as the space of $\mathbb{T}$-valued points of a tropicalization in Lorscheid's sense, and this gives rise to a natural map from the Berkovich analytification to the Kajiwara-Payne tropicalization — however, I have yet to familiarize myself with the details. Indeed, there are a variety of other notions of tropicalization that can be placed on this footing, but I will make no attempt to explicate them here — see [2] for the gory details. I will, however, note that Lorscheid's definitions, stated in full generality, actually allow for bases other than $\mathrm{Spec}\, \mathbb{T}$ and therefore encompass, for example, the Foster-Ranganathan analytification and tropicalization arising from a higher-rank valuation.

# References


[1] Oliver Lorscheid. Blueprints and tropical scheme theory. *Unpublished lecture notes: http://oliver.impa.br/2018-Blueprints/lecturenotes.pdf*, 2018.

[2] Oliver Lorscheid. Scheme theoretic tropicalization. *Preprint arXiv:1508.07949v3*, 2019.