The Topology of Milnor Fibers

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December 15, 2020

Outline of the talk

I. Introduction
II. Results for nice polynomials
III. Slices and equisingularity results
IV. Future directions

I. Introduction

Milnor(-Lê) fibration

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial vanishing at the origin. Then, for S_ε a small enough sphere around the origin and $K_\varepsilon = V(f) \cap S_\varepsilon$ the link,

$$\frac{f}{|f|} : S_\varepsilon \setminus K_\varepsilon \to S^1$$

is a smooth locally trivial fibration, called the **Milnor fibration** of f at the origin. The **Milnor fiber** F_f is a parallelizable $2n$-manifold, and in the case of an isolated singularity it is bounded by the link.

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Equivalently, if $\delta > 0$ is sufficiently small relative to ε, we can consider the restriction

$$f : B_\varepsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

for B_ε the ball around the origin and D^*_δ the punctured disk in $\mathbb{C}$.

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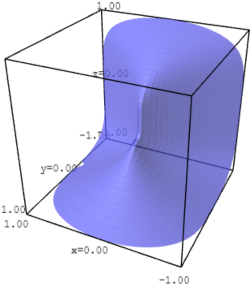
Equivalently, if $\delta > 0$ is sufficiently small relative to ε, we can consider the restriction

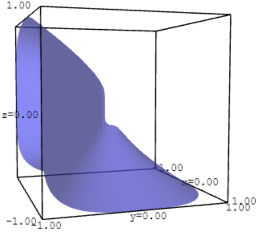
$$f : \bar{B}_\varepsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

for $\bar{B}_\varepsilon$ the ball around the origin and D_δ^* the punctured disk in $\mathbb{C}$.

Example: A Brieskorn-Pham singularity

Consider $f(x, y, z) = x^2 + y^3 + z^5$:





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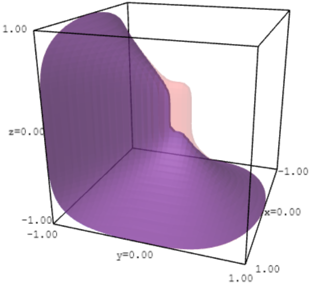

Some generalizations of this setup

There are also more general versions of the Milnor fibration:

- We can consider non-polynomial complex-analytic functions $f : \mathbb{C}^{n+1} \to \mathbb{C}$.
- We can consider complete intersection singularities instead of hypersurface singularities using functions $f : \mathbb{C}^{n+k} \to \mathbb{C}^k$.
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- We can consider functions $f : X \to \mathbb{C}$ on possibly singular spaces X.
- There is a sheaf-theoretic perspective which lets us generalize questions about the cohomology of the fiber and can be used to obtain some results even in the classical case.

Some applications and connections with other areas

Milnor fibers and their cohomology show up in various nearby areas of math, including:

- The proof of the Weil conjectures
- Symplectic geometry
- Enumerative geometry
- Birational geometry
- The theory of characteristic classes for singular varieties
- Hodge theory

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On the applied side, they're involved in nearest-point problems by way of the Euclidean distance degree, which has been used to study computer vision and chemical reaction networks, among other things.

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In all but the simplest cases, the computation of the cohomology of the Milnor fiber F_f is still extremely open. For example:

- If $V(f)$ is a central hyperplane arrangement, it is not known whether the Betti numbers of F_f are combinatorially determined.
- (Bobadilla's conjecture) If f has a 1-dimensional critical locus Σ_f with a single irreducible component, it is not known whether the cohomology of the Milnor fiber at a point where Σ_f is itself singular can be the same as the cohomology of the Milnor fiber at a generic point of Σ_f.

II. Results for nice polynomials

The critical locus and homology

- If s is the dimension at the origin of f's critical locus, then F_f is $(n-s-1)$-connected, and in particular the reduced homology is zero outside of the interval $[n-s, n]$.

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- In the case of an isolated singularity ($s=0$), we also have that $\tilde{H}_n(F_f) \cong \mathbb{Z}^{\mu_f}$, where

$$\mu_f = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right)}$$

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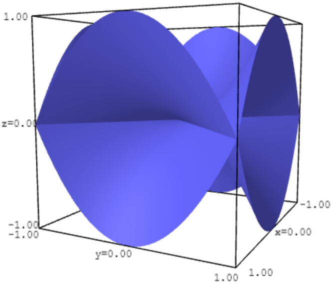
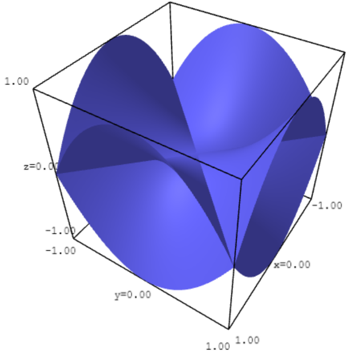
  is the **Milnor number** of f at the origin.
- In fact, F_f is homotopy equivalent to a bouquet of μ_f n-spheres in this case.

Examples

- If $f(x,y,z) = x^2 + y^3 + z^5$, F_f is a bouquet of 8 2-spheres.
- Consider $f(x,y,z) = x^2y^2 + z^2$, and note that the critical locus $V(xy^2, x^2y, z)$ has dimension $s = 1$ at the origin. It turns out that in this case $F_f \simeq S^1 \vee S^2 \vee S^2$.





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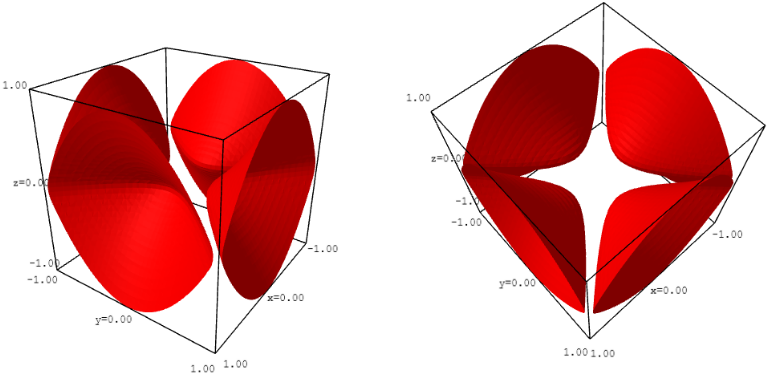

The (weighted) homogeneous case

Suppose $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is (weighted) homogeneous of degree d, so that $V(f)$ is the affine cone over the hypersurface cut out by f in (weighted) projective n-space.

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- Then the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus V(f)$ gives a local trivialization for f over $\mathbb{C} \setminus \{0\}$; in this case f's restriction is called the **global Milnor fibration**.
- This is fiber diffeomorphic to the usual Milnor fibration. In particular, F_f is diffeomorphic to $f^{-1}(1)$.

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- This is fiber diffeomorphic to the usual Milnor fibration. In particular, F_f is diffeomorphic to $f^{-1}(1)$.
- By considering the quotient of $f^{-1}(1)$ under the action of the dth roots of unity, we can see that the Milnor fiber is a d-fold cover of the complement of the hypersurface cut out by f in (weighted) projective n-space.

Combining polynomials

Theorem (Generalized Thom-Sebastiani; Nemethi 1991)

Let $p : \mathbb{C}^2 \to \mathbb{C}$, $g : \mathbb{C}^m \to \mathbb{C}$, and $h : \mathbb{C}^n \to \mathbb{C}$ be polynomials, and define $f : \mathbb{C}^{m+n} \to \mathbb{C}$ by $f = p(g, h)$. Then F_f is given up to homotopy by starting with the total space of a fiber bundle with base F_p and fiber $F_g \times F_h$, then attaching some appropriate numbers of copies of $cF_g \times F_h$ and $F_g \times cF_h$.

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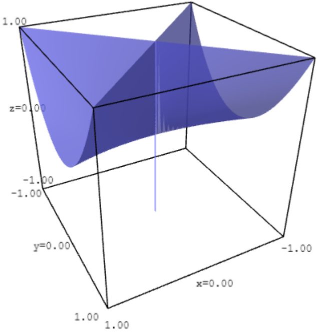
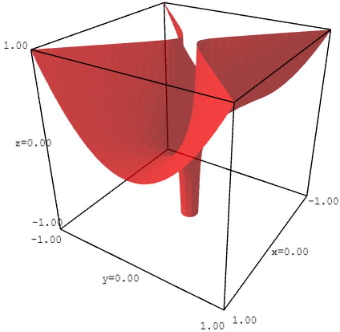
Corollary (Classical Thom-Sebastiani; Sakamoto 1974)

$F_{g+h} \simeq F_g * F_h$, since in this case $p(c, d) = c + d$ has trivial Milnor fiber and the numbers mentioned above are both 1.

Example: Whitney umbrella

$f(x, y, z) = y^2 - zx^2$ has critical locus $V(x^2, xz, y)$ (the z-axis plus some fuzz at the origin), and we can see by Thom-Sebastiani that $F_f \simeq F_{y^2} * F_{zx^2} \simeq \{2 \text{ pts}\} * \{zx^2 = 1\} \simeq S\{z = 1/x^2\} \simeq SS^1 \simeq S^2$.

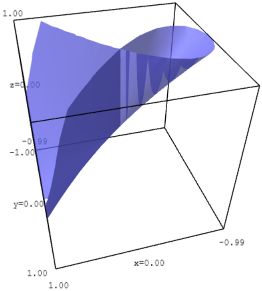


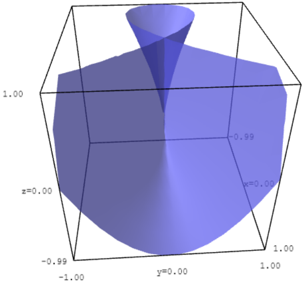


III. Slices and equisingularity results

Example: The Whitney umbrella as a family

With a slight change of coordinates ($t = z - x$), the defining function of the Whitney umbrella can be written as $f(t, x, y) = y^2 - x^3 - tx^2$;





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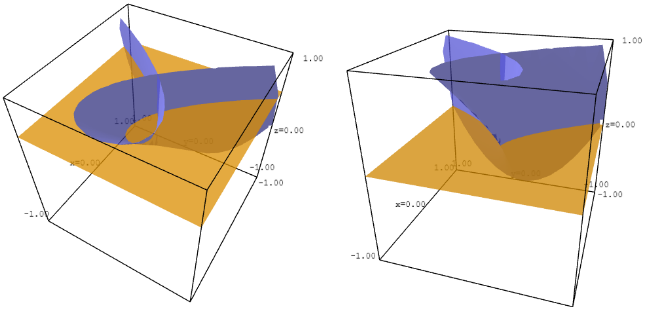
With a slight change of coordinates ($t = z - x$), the defining function of the Whitney umbrella can be written as $f(t, x, y) = y^2 - x^3 - tx^2$; slicing by the hyperplanes $t = k$ gives us polynomials $f_t(x, y)$ such that $V(f_t)$ is a family of nodal plane curves degenerating to a cusp at $t = 0$.



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    - Lê and Ramanujan (mostly) answered this for a family of isolated singularities with constant Milnor number.
- How do the Milnor fibers of f at different points in the critical locus relate to each other?

Lê's attaching result

Definition

Let x_0 be a sufficiently generic linear form on $\mathbb{C}^{n+1}$. Then the **relative polar curve** *of f at the direction defined by x_0 is defined as the closure of the locus where the Jacobian matrix of $(f, x_0) : \mathbb{C}^{n+1} \to \mathbb{C}^2$ has rank exactly 1 and denoted by Γ^1_{f,x_0}.*

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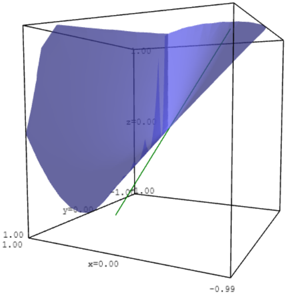
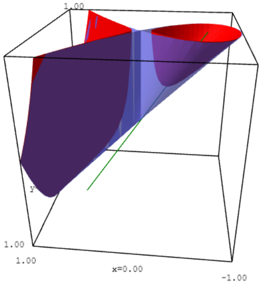
Theorem (Lê 1973)

In the above setting, if we let $H = V(x_0)$, then F_f is given up to homotopy by attaching $(V(f) \cdot (\Gamma^1_{f,x_0})_{red})_0$ n-cells to $F_{f|_H}$.

Recurring example: Whitney umbrella

Let $f(t,x,y) = y^2 - x^3 - tx^2$ as before. Then
$\Gamma^1_{f,t} = \overline{\{-x^2 \neq 0, -3x^2 - 2tx = 0, 2y = 0\}} = V(3x+2t, y)$,





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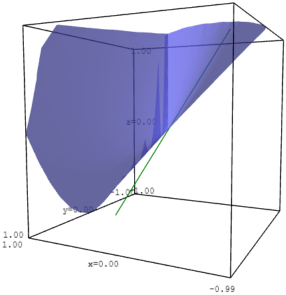

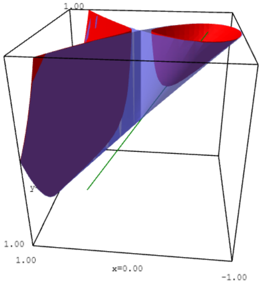

Polar varieties

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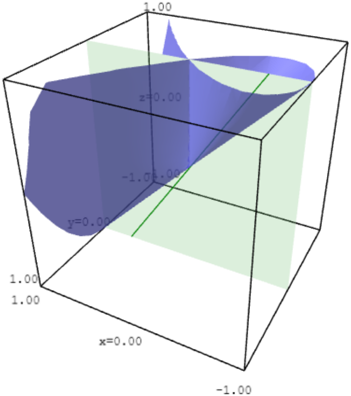
Definition

Let $\vec{x} = (x_0, \ldots, x_n)$ be a sufficiently generic coordinate system for $\mathbb{C}^{n+1}$. Then the kth **(relative) polar variety** $\Gamma^k_{f,\vec{x}}$ *of f with respect to $\vec{x}$ is defined to be the closed subscheme of $\mathbb{C}^{n+1}$ obtained by starting with $V(\frac{\partial f}{\partial x_k}, \ldots, \frac{\partial f}{\partial x_n})$ and removing all components contained in the critical locus $V(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$.*

Example: Whitney umbrella

Let $\vec{x} = (t, x, y)$ and $f(t, x, y) = y^2 - x^3 - tx^2$. Then $\Gamma^0_{f,\vec{x}} = \emptyset$, we have already seen $\Gamma^1_{f,\vec{x}} = V(3x + 2t, y)$, and $\Gamma^2_{f,\vec{x}} = V(y)$.



Lê cycles and numbers

Definition

If $\vec{x}$ is sufficiently generic as before, the kth **Lê cycle** *of f with respect to $\vec{x}$ is denoted by $[\Lambda^k_{f,\vec{x}}]$ and defined to be equal to the cycle $[\Gamma^{k+1}_{f,\vec{x}} \cap V(\frac{\partial f}{\partial x_k})] - [\Gamma^k_{f,\vec{x}}]$.*

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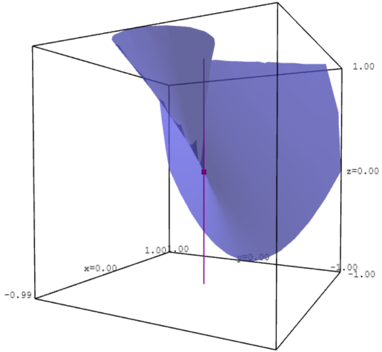
As long as the intersection of $[\Lambda^k_{f,\vec{x}}]$ with $[V(x_0, \ldots, x_{k-1})]$ is zero-dimensional at the origin, we call the intersection number $([\Lambda^k_{f,\vec{x}}] \cdot [V(x_0, \ldots, x_{k-1})])_0$ the kth **Lê number** *of f with respect to $\vec{x}$ at the origin and denote it by $\lambda^k_{f,\vec{x}}(0)$.*

Example: Whitney umbrella

$\vec{x} = (t, x, y)$, $f(t, x, y) = y^2 - x^3 - tx^2$. Then
$[\Lambda^0_{f,\vec{x}}] = [V(3x + 2t, y) \cap V(x^2)] - [\emptyset] = [V(x^2, 3x + 2t, y)] = 2[0]$
and $[\Lambda^1_{f,\vec{x}}] = [V(y) \cap V(3x^2 + 2tx)] - [V(3x + 2t, y)] = [V(x, y)]$.
We have $\lambda^0_{f,\vec{x}}(0) = 2$ and $\lambda^1_{f,\vec{x}}(0) = ([V(x, y)] \cdot [V(t)])_0 = 1$.



Some facts

- The underlying sets of the Lê cycles are contained in the critical locus, as claimed, and in fact the union of the underlying sets for the $0, \dots, k$th Lê cycles is the part of the underlying set of $\Gamma^{k+1}_{f,\vec{x}}$ that lies in the critical locus.

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Theorem (Massey 1990)

In the setting of the previous definition, F_f has a cell decomposition with $\lambda^k_{f,\vec{x}}(0)$ $(n-k)$-cells for $0 \le k \le s$ (plus one extra 0-cell to start things off).

Constancy of the fibration

Theorem (Massey 1995)

If f_t is a family of polynomials such that the Lê numbers of f_t at the origin are constant in t, and our choice of coordinates is sufficiently generic with respect to the family, then the homology of the fibers F_{f_t} is independent of t.

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Proof idea

For large enough j, $f_t + {x_0}^j$ has Lê numbers calculable from those of f_t (Massey's "Lê-Yomdin formulas"), hence independent of t, and a critical locus of dimension $s-1$. By induction, it is then enough to argue that the constancy of the homology of $f_t + {x_0}^j$ implies that of the homology of $f_t + w^j$ for some new variable w, and that this in turn implies that of the homology of f_t.

IV. Future directions

Does quantifying the scheme structure of the critical locus tell us anything?

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Let $(X, \mathcal{O}_X)$ be a (locally Noetherian) scheme. Then, for each point $x \in X$ (not necessarily closed), the length of the ideal saturation $(0) : \mathfrak{m}_x{}^\infty$ in $\mathcal{O}_{X,x}$, which we'll call $\mathfrak{a}_x(X)$, should measure "the extent to which x is an associated point of X". For example:

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- $\mathfrak{a}_x(X) > 0$ if and only if x is associated.
- If X is an integral scheme with generic point η, $\mathfrak{a}_\eta(X) = 1$ and $\mathfrak{a}_x(X) = 0$ for all other points $x \in X$.
- If X is irreducible with generic point η, $\operatorname{mult}_x(X) = \mathfrak{a}_\eta(X) \operatorname{mult}_x(X_{\text{red}})$.

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Let $(X, \mathcal{O}_X)$ be a (locally Noetherian) scheme. Then, for each point $x \in X$ (not necessarily closed), the length of the ideal saturation $(0) : \mathfrak{m}_x{}^{\infty}$ in $\mathcal{O}_{X,x}$, which we'll call $\mathfrak{a}_x(X)$, should measure "the extent to which x is an associated point of X". For example:

- $\mathfrak{a}_x(X) > 0$ if and only if x is associated.
- If X is an integral scheme with generic point η, $\mathfrak{a}_\eta(X) = 1$ and $\mathfrak{a}_x(X) = 0$ for all other points $x \in X$.
- If X is irreducible with generic point η, $\operatorname{mult}_x(X) = \mathfrak{a}_\eta(X)\operatorname{mult}_x(X_{\mathrm{red}})$.
- If $X = V(xy, y^{k+1})$ in $\mathbb{C}^n$, then $\mathfrak{a}_{(0)}(X) = 1$, $\mathfrak{a}_{(x,y)}(X) = k$, and $\mathfrak{a}_x(X) = 0$ for all other points $x \in X$.

Does quantifying the scheme structure of the critical locus tell us anything?

Question

Let $X = \operatorname{Spec} \mathcal{O}_{\mathbb{C}^{n+1},0}/(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_0})$ *be the scheme-theoretic critical locus of* f *at the origin. Given the associated points* p *of* X*, the values* $\mathfrak{a}_p(X)$*, and the dimensions and multiplicities of the components* $C_p = \overline{\{p\}}$ *at the origin, how much can we say about the Betti numbers of* F_f *at the origin? What can we say about other equisingularity questions?*

(It's possible we want X *to be the spectrum of the completion or henselization of this ring instead.)*

Does quantifying the scheme structure of the critical locus tell us anything?

One idea for a way to attack this:
The germ of f at the origin corresponds to the map

$$\mathcal{O}_{\mathbb{C}^{n+1},0} = \mathbb{C}[x_0,\ldots,x_n]_{(x_0,\ldots,x_n)} \xleftarrow{\phi} \mathbb{C}[t]_{(t)} = \mathcal{O}_{\mathbb{C},0}$$

given by $\phi(t) = f$. We can pass to a map $(\mathcal{O}^h_{\mathbb{C},0})_t \to (\mathcal{O}^h_{\mathbb{C}^{n+1},0})_f$, and a comparison theorem from SGA tells us that the étale cohomology of

$$\mathrm{Spec}\left((\mathcal{O}^h_{\mathbb{C}^{n+1},0})_f \otimes_{(\mathcal{O}^h_{\mathbb{C},0})_t} \overline{(\mathcal{O}^h_{\mathbb{C},0})_t}\right)$$

gives the singular cohomology of the Milnor fiber (when the coefficients are finite).

Thanks for watching!