Milnor Fiber Consistency via Flatness

by

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Abstract

The Milnor fibration, which was introduced by Milnor in [Mil68], captures the local behavior of a holomorphic function near a critical point and has been the subject of much research in the subsequent decades. Despite this, the answers to many basic questions about its topology remain poorly understood. The results of the author in [Hof] suggest that the solution to this problem should lie in analysis of the embedding in the ambient space of the function's critical locus, which we endow with the structure of a complex-analytic space by regarding it as the vanishing of the function's Jacobian ideal. Here we will explicate these results after building up the necessary background machinery on complex-analytic geometry, stratification theory, and the Milnor fibration itself.

Dedication

To the friends and family who have supported me through hard times, and most of all to my parents, to whom I owe more than I can say.

Declaration

I declare that the contents of this thesis reflect my own original research, except where otherwise indicated, and that all sources used have been properly cited. I further declare that I have not submitted the work contained herein as a thesis or similar document to any academic institution other than the University of Wisconsin-Madison, or in pursuit of any other degree.

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Chapter 1

Introduction

Consider a holomorphic function $f : U \to \mathbb{C}$ defined on a neighborhood of the origin in $\mathbb{C}$, and suppose first that the partial derivatives of f do not all vanish at the origin. Then, by standard facts from differential geometry, f can locally be written as a coordinate projection, so that the restriction of f to an open ball around the origin is a trivial fibration, with fiber a smaller-dimensional ball, over values near $f(0)$; that is, we have the local picture depicted in Figure 1.1.

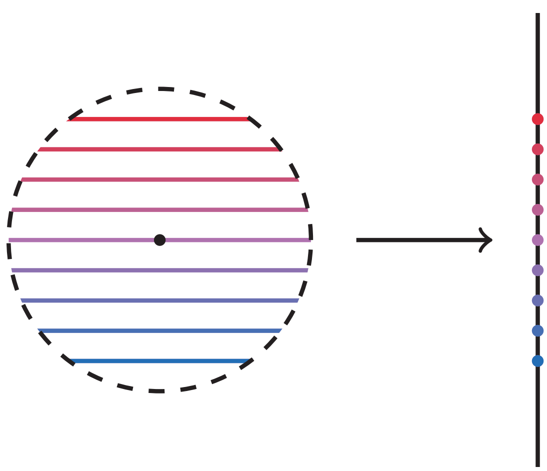


Figure 1.1: A local coordinate projection — e.g., $f(x, y) = y$.

If the partial derivatives of f at the origin do all vanish — that is, if the origin is a **critical point** of f — the situation becomes more complicated. The fiber of f through such a point is singular, at least as a **complex-analytic space** (for which see Section

2.1), while the fibers over all nearby values of f will locally turn out to be smooth. Hence, in this case, the restriction of f near the origin cannot define a fiber bundle over any neighborhood of $f(0)$ as it does in the previous situation. However, it will turn out that throwing away the fiber through the origin solves this problem — we obtain a smooth locally trivial fibration over a punctured neighborhood of $f(0)$, called the **Milnor fibration**, one example of which is depicted in Figure 1.2.

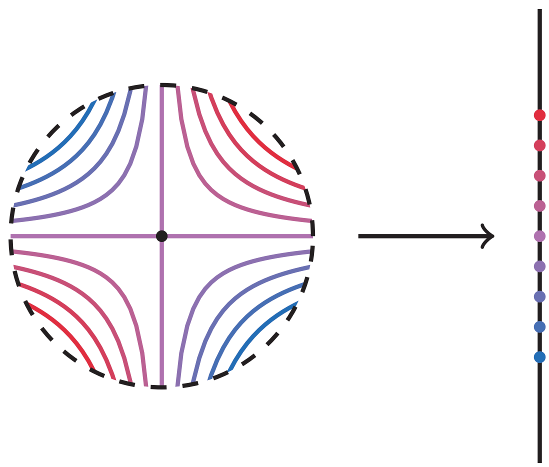


Figure 1.2: Local fibers around a critical point — e.g., the origin for $f(x, y) = xy$.

The fibers of this fibration are simply the parts near the critical point of nearby smooth fibers of f, and it is natural to wonder whether the topology of its fibers and the monodromy of the fibration as whole can be computed from f — that is, **how much we know about the way a given holomorphic function behaves locally at a point of its domain**. As we will see in Section 4.1, this is reasonably well-understood in the case of an isolated critical point. However, outside these circumstances there is much that remains to be discovered — many basic facts about the topology of the Milnor fibration are still unknown in general.

The remainder of this thesis will focus on the idea, proposed by the author in [Hof], that such facts should be recoverable from the **Jacobian ideal** generated by the partial derivatives of f in the local ring of holomorphic function germs at the critical point — that is, **information about the Milnor fibration should be captured by the structure of the critical locus as a subscheme or complex-analytic subspace of**

the ambient space.

We will begin in Chapter 2 by introducing the machinery of schemes and complex-analytic spaces and surveying some basic facts from algebraic and complex-analytic geometry. In particular, we will focus on developing and motivating the notion of a **normal cone**, which plays a key role in the results of [Hof], and explaining how this construction can be used to provide geometric interpretations of key concepts from algebraic geometry,

In Chapter 3, we will examine the **singularities** of complex-analytic spaces and explore the machinery of **stratifications**, which allows us to study singular spaces by breaking them into smooth pieces. This will culminate in the first of the main results of [Hof], a stratification theorem for families of holomorphic functions based on the flatness over the parameter spaces of the normal cones to their critical loci.

Finally, in Chapter 4, we will introduce the Milnor fibration itself, proving its existence and surveying some of the main techniques for controlling its behavior in families. This will conclude with a summary of the results of [Hof], which again allow us to do so in terms of the flatness of normal cones to critical loci.

Chapter 2

Normal Cones in Algebraic and $\mathbb{C}$-Analytic Geometry

Here we lay the foundations for the results of later chapters by introducing many of the fundamental objects we will work with and establishing basic facts about them. Our main subjects will be algebraic geometry, complex-analytic geometry, and the relationship between the two — the overarching goal in discussing these will be to furnish the reader with geometric intuitions for all constructions and results, even those which are usually presented in more algebraic terms.

To this end, we will work toward the definition of a particular object, the **normal cone** of a closed subspace, which is particularly useful in this regard. Section 2.1 will remind the reader of some salient details from the algebraic and complex-analytic settings, and Section 2.2 will build on this by explicating the notion of a **cone** in algebraic or complex-analytic geometry, providing a geometric interpretation of coherent sheaves along the way. Section 2.3 will introduce the definition of the normal cone itself and establish some basic geometric intuitions about it, while Section 2.4 will conclude by exploring some of the applications of this construction in geometrically interpreting common objects and facts from algebraic geometry.

2.1 Schemes and $\mathbb{C}$-Analytic Spaces

We begin by recalling some standard notions from algebraic and complex-analytic geometry. Since these are generally well-known, we will not aim to give a detailed introduction, only to provide a broad overview of the intuitions involved and point the reader in the direction of more detailed explications.

2.1.1 Basic Constructions

Perhaps the most fundamental objects in algebraic geometry are the **schemes**, spaces which can be formalized in a number of ways but which are essentially based on the following idea. If R is a ring, then there are compelling analogies of quotient maps $R \twoheadrightarrow R/I$ and localization maps $R \to R_f$ to inclusions in the other direction of closed and open subsets respectively into a topological space; hence we can think of rings as geometric objects of some kind simply by formally reversing the direction we think of maps between them as having. These objects are the **affine schemes**, and from them we recover schemes more generally simply by allowing ourselves to glue things together along open covers, as we can in the case of topological spaces. The affine scheme $\operatorname{Spec} R$ associated to a ring R is called its **spectrum**, and most typically thought of as the topological space of prime ideals of R (with the topology induced by localizations as above) together with a **structure sheaf** $\mathcal{O}_{\operatorname{Spec} R}$ capturing the extra information not contained in the space itself — for the details of these constructions and a more comprehensive account of the subject, see, e.g., [Har77; Vak23].

In the case of a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over an algebraically closed field $\mathbb{K}$, Hilbert's Nullstellensatz, for which see, e.g., Theorem I.1.3A of [Har77], allows us to identify the closed points of $\operatorname{Spec} R$ with elements of $\mathbb{K}^n$, and hence likewise for those of locally closed subschemes of $\operatorname{Spec} R$. In the case where $\mathbb{K} = \mathbb{C}$ is the field of complex numbers, the existence of the classical topology on $\mathbb{C}^n$ then lets us visualize these schemes, and finite-type $\mathbb{C}$-schemes more generally, in terms of polynomially-defined subsets of the familiar Euclidean space, although care must be taken due to the differences between the topologies involved and the existence of non-closed points.

To bridge the gap between these two perspectives, we recall the notion of a complex-analytic space. We can consider on the topological space $\mathbb{C}^n$ the sheaf of holomorphic functions in n variables; just as schemes can be thought of as locally ringed spaces which are locally isomorphic to affine schemes, then, we can consider locally ringed spaces which are locally isomorphic to subspaces cut out from open subsets of $\mathbb{C}^n$ by the vanishing of collections of holomorphic functions together with the corresponding quotients of this sheaf. Such a space will be called a **complex-analytic space**. In particular, we can then construct for any finite-type $\mathbb{C}$-scheme X the corresponding complex-analytic space X^{an}, called its **analytification**, as well as a natural map $X^{\mathrm{an}} \to X$ of locally ringed spaces — for further exposition of these notions, see [Ser56; Nar66; GR65; Fis76; Har77]. Note that some authors, such as Fischer ([Fis76]), require a complex-analytic space to be Hausdorff, which means that the analytification exists only for finite-type $\mathbb{C}$-schemes which are moreover separated, but we will not take this requirement to be part of the definition.

We note that complex-analytic spaces share an important property with locally Noetherian schemes, by the following **Oka coherence theorem**:

Theorem 2.1.1 (e.g., Theorem IV.3 of [Nar66]). *Let X be a complex-analytic space. Then $\mathcal{O}_X$ is coherent as a sheaf of $\mathcal{O}_X$-modules.*

2.1.2 Local Study of Schemes and $\mathbb{C}$-Analytic Spaces

In studying both schemes and complex-analytic spaces, particularly from the perspective of singularities, it will often be of interest to work locally around a single point, typically a closed one — that is, to consider only the **germ** of the space around that point, which is the object given by working with the equivalence classes of functions, spaces, and other objects defined over the space up to the equivalence relation given by agreement on any open neighborhood of the point. This can be regarded as a kind of "formal intersection of all open neighborhoods of x". In the algebraic case, fortunately enough, this object itself exists as a scheme, the spectrum of the local ring at the point in question, and so we can readily study germs of schemes without needing to develop new machinery specialized to the task.

The definition of a complex-analytic space, on the other hand, is not flexible enough for the corresponding analytic statement to be true. Instead, however, we have the following result:

Proposition 2.1.2 (e.g., Section 0.21 of [Fis76])**.** *The contravariant functor from the category of germs of complex-analytic spaces to the category of quotients of convergent power series rings over $\mathbb{C}$ given by taking local rings is an anti-equivalence of categories.*

Thus germs of complex-analytic spaces can in some respects be regarded simply as affine schemes of a particular kind. However, this identification breaks down somewhat as soon as any kind of non-locality is introduced — perhaps most notably, for example, the tensor product of two convergent power series rings over a third need not itself be a convergent power series ring, and so no notion of a fiber product of germs of complex-analytic spaces can be recovered straightforwardly from the usual fiber product of schemes. Hence some care needs to be taken in interpreting complex-analytic space germs in this way; nevertheless, this perspective remains useful in dealing with such objects without needing to reprove analogues of existing results from scheme theory.

If X is a finite-type $\mathbb{C}$-scheme, more can be said about the relationship between the germ of X at a closed point and that of its analytification, given some additional machinery. We begin by noting that, for schemes in general, we have a number of ways to "zoom in further" on a point beyond simply taking the germ:

Definition 2.1.3 (e.g., [Har77; Eis04])**.** Let X be a scheme, $x \in X$ a point, $R = \mathcal{O}_{X,x}$ the local ring of X at x, and $\mathfrak{m}$ its maximal ideal.

- For each $k \geq 0$, we define the **kth-order infinitesimal neighborhood** of x in X to be the affine scheme $\operatorname{Spec} R/\mathfrak{m}^{k+1}$.

- We define the **completion** $\hat{R}$ of R to be the inverse limit of the diagram $\cdots \to R/\mathfrak{m}^3 \to R/\mathfrak{m}^2 \to R/\mathfrak{m} \to 0$ of rings.

Then we have natural maps as follows:

$$x = \operatorname{Spec} R/\mathfrak{m} \hookrightarrow \operatorname{Spec} R/\mathfrak{m}^2 \hookrightarrow R/\mathfrak{m}^3 \hookrightarrow \cdots \hookrightarrow \operatorname{Spec} \hat{R} \to \operatorname{Spec} R \hookrightarrow X$$

We can view this as giving a kind of hierarchy of "neighborhoods of x in X" — thinking of the elements of R as the germs of algebraic functions on X at x, we can regard the kth-order infinitesimal neighborhood as "a neighborhood of x so small that restricting a function to it retains only the **k-jet**, the information of the partial derivatives of order $\leq k$ at x". $\operatorname{Spec} \hat{R}$, being in some sense the "union" of all these infinitesimal neighborhoods, then carries exactly the information of all possible choices of derivatives of all orders at x, and so we can think of it as "a neighborhood so small that every formal power series expansion converges". There is also another local ring, the **henselization** of R, whose spectrum sits between $\operatorname{Spec} \hat{R}$ and $\operatorname{Spec} R$, but we will not discuss it here.

If X is a finite-type $\mathbb{C}$-scheme and x a closed point, we have $R = \mathbb{C}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/I$ for some ideal I — $\hat{R}$ is then the quotient of the formal power series ring $\mathbb{C}[[x_1, \ldots, x_n]]$ by the corresponding ideal. The local ring $\mathcal{O}_{X^{\mathrm{an}},x^{\mathrm{an}}}$ of X^{an} at the point x^{an} sent to x by the map $X^{\mathrm{an}} \to X$, similarly, is given by the corresponding quotient of $\mathbb{C}\{x_1, \ldots, x_n\}$, the ring of power series which converge on some neighborhood of the origin in $\mathbb{C}^n$ with the analytic topology. We can then see that the affine scheme corresponding to the germ of X^{an} at x sits between $\operatorname{Spec} \hat{R}$ and $\operatorname{Spec} R$ in the above hierarchy — that is, on the level of germs at a closed point the analytification of a finite-type $\mathbb{C}$-scheme can be thought of simply as a particular kind of scheme-theoretic "neighborhood" of the point whose definition arises from the classical topology on $\mathbb{C}$.

It is worth clarifying what exactly we mean by "neighborhood" in the case of each of these objects. The spectrum of the local ring is, on the level of the underlying topological spaces, contained in X — however, it need not literally be a neighborhood in the traditional topological sense because it does not necessarily contain any open subset around x. The infinitesimal neighborhoods are likewise set-theoretically subspaces of X — indeed, the underlying space of each is simply the point x — but they are Zariski-closed subschemes rather than open ones and share few formal similarities with topological neighborhoods, despite the name. The spectra of the local ring's completion and, in the case where it exists, the analytification's local ring, on the other hand, are not necessarily even inclusions on the level of topological spaces, though they are so on closed points. However, the maps from these two schemes and from the spectrum of the local ring to X share an important property with open inclusions:

Definition 2.1.4 (e.g., [Har77]). A map $\phi : X \to Y$ of schemes or complex-analytic spaces is said to be **flat** if, for each point $p \in X$, the induced map $\mathcal{O}_{Y,\phi(p)} \to \mathcal{O}_{X,p}$ of local rings makes $- \otimes_{\mathcal{O}_{Y,\phi(p)}} \mathcal{O}_{X,p}$ an exact functor.

The maps from the aforementioned affine schemes to the ambient scheme are all flat, and as alluded to the same is true for inclusions of open subschemes. Intuitively, flat maps are those with fibers which are "consistent" in some sense, and hence flatness is typically taken as the correct notion of what it should mean for a morphism to define a "family" or "deformation" of its fibers over the target space. We will make this notion precise in Subsection 2.4.1, once we have developed the necessary machinery.

To conclude our discussion of the local relationship between a finite-type $\mathbb{C}$-scheme and its analytification, we introduce also the following notion:

Definition 2.1.5 (e.g., [Mat89; Vak23]). A map of schemes is said to be **faithfully flat** if it is flat and surjective.

This turns out to be equivalent to the requirement that pullback along the map is not only exact, as in the case of flatness, but moreover does not make exact any sequence of sheaves of modules which was not already exact. Note that, by, e.g., Theorem 7.2 of [Mat89], the surjectivity can be checked only on closed points so long as the map in question is flat. For our purposes, the usefulness of this notion lies in the following fact:

Proposition 2.1.6. *$\mathbb{C}$-algebra maps of the form $\mathbb{C}\{x_1, \ldots, x_a\}[y_1, \ldots, y_b]_{(x_1,\ldots,x_a,y_1,\ldots,y_b)} \to \mathbb{C}\{x_1, \ldots, x_a, y_1, \ldots, y_b\}$ are faithfully flat.*

In particular, let X be a finite-type $\mathbb{C}$-scheme and $x \in X$ a closed point. Then, if x^{an} is the point of X^{an} mapped to x by $X^{an} \to X$, the natural map $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{an},x^{an}}$ is faithfully flat.

We defer the proof of this result until Subsection 2.4.1; note however, that the first half follows from the second since faithful flatness is preserved under pullback (e.g., [Mat89]). The result's practical utility lies mainly in computing examples; in situations where we are interested in verifying facts about complex-analytic spaces which arise as analytifications of finite-type $\mathbb{C}$-schemes, the proposition will be used to show that it is permissible to carry out the computations of interest algebraically.

2.1.3 Spaces from Sheaves of Algebras

We have already mentioned that, by taking the spectrum of a ring, we can reinterpret it as a geometric object instead of an algebraic one. We now expand this to objects which are partially algebraic and partially geometric: sheaves of algebras over a scheme.

Definition 2.1.7 (e.g., [Vak23])**.** Let X be a scheme and $\mathcal{A}$ a quasicoherent sheaf of $\mathcal{O}_X$-algebras. Then the **relative spectrum** of $\mathcal{A}$, which we denote by $\underline{\operatorname{Spec}}\,\mathcal{A}$, is the scheme over X obtained by patching together the maps $\operatorname{Spec}\mathcal{A}(U_\alpha) \to U_\alpha$ for $\{U_\alpha\}$ an affine open cover of X.

Note that, by the definition, $\underline{\operatorname{Spec}}\,\mathcal{A} \to X$ is an affine morphism — in fact, it is not difficult to show that $\underline{\operatorname{Spec}}(-)$ gives an anti-equivalence between the category of quasicoherent sheaves of $\mathcal{O}_X$-algebras and the category of schemes affine over X, with quasi-inverse given by pushforward of the structure sheaf. The independence of the choice of cover can be verified by showing that $\underline{\operatorname{Spec}}\,\mathcal{A}$ represents the functor of schemes over X which takes $\pi : Y \to X$ to $\operatorname{Hom}_{\mathcal{O}_Y}(\pi^*\mathcal{A}, \mathcal{O}_Y)$ — see, e.g., Exercise II.5.17 of [Har77], or Section 17.1 of [Vak23].

There is an analogue of this construction in the analytic setting — however, since complex-analytic spaces are less versatile than schemes, it requires additional restrictions on the sheaves of algebras involved.

Definition 2.1.8 ([Hou61])**.** Let X be a complex-analytic space and $\mathcal{A}$ a sheaf of $\mathcal{O}_X$-algebras which is **finitely presented** — that is, it is locally given as a quotient of a polynomial algebra sheaf in finitely many variables over $\mathcal{O}_X$ by finitely many sections. Then the **analytic spectrum** of $\mathcal{A}$, which we denote by $\operatorname{Specan} \mathcal{A}$, is the complex-analytic space over X which is given on any open subscheme U with $\mathcal{A}|_U \cong \mathcal{O}_U[x_1, \ldots, x_k]/(f_1, \ldots, f_\ell)$ by the complex-analytic vanishing of the holomorphic functions $f_1, \ldots, f_\ell$ in $U \times \mathbb{C}^k$.

As before, this can be verified to be well-defined by showing that the complex-analytic space thus constructed for a particular open cover represents the appropriate functor of complex-analytic spaces over X — see [Hou61] for the details.

We conclude by noting that, in the case of a quasicoherent sheaf of algebras which is moreover $\mathbb{N}$-graded, we can make use of the additional structure to define an additional object. The following construction is familiar from algebraic geometry:

Definition 2.1.9 (e.g., [Vak23])**.** Let R be an $\mathbb{N}$-graded ring. Then the **homogeneous spectrum** of R is the scheme $\operatorname{Proj} R$ over $\operatorname{Spec} R_0$ whose underlying set is the set of all homogeneous prime ideals of R not containing $R_{\geq 1}$, with topology generated by the subsets consisting of all such elements not containing a given homogeneous element of R and structure sheaf given by assigning to each such set the degree-zero part of the corresponding localization of R.

In the case where R is a quotient of a polynomial ring by a homogeneous ideal, we can think of $\operatorname{Proj} R$ as the "space of lines through the origin in $\operatorname{Spec} R$" — more generally, we can think of $\operatorname{Proj} R$ as being given by removing $\operatorname{Spec}(R/R_{\geq 1})$ from $\operatorname{Spec} R$ and taking the quotient by an action of the multiplicative group arising from the grading, an idea we will revisit in more detail when we define the analogous analytic notion in Subsection 2.2.4.

As with the usual spectrum, we can define a relative version:

Definition 2.1.10 (e.g., [Vak23])**.** Let X be a scheme and $\mathcal{A}$ a quasicoherent sheaf of $\mathbb{N}$-graded $\mathcal{O}_X$-algebras. Then the **relative homogeneous spectrum** of $\mathcal{A}$, which we denote by $\underline{\operatorname{Proj}}\,\mathcal{A}$, is the scheme over X obtained by patching together the maps $\operatorname{Proj}\mathcal{A}(U_\alpha) \to U_\alpha$ for $\{U_\alpha\}$ an affine open cover of X.

The morphisms to X thus obtained when $\mathcal{A}$ is moreover finitely generated in degree 1 over $\mathcal{O}_X$, called the **projective morphisms**, are of great importance in algebraic geometry and satisfy many useful properties — for details, see, e.g., Section II.7 of [Har77] or Chapter 17 of [Vak23]. As mentioned, we defer the discussion of the analogous notion for graded sheaves over a complex-analytic space to Subsection 2.2.4.

2.2 Vector Bundles, Linear Fiber Spaces, and Cones

As discussed in Section 2.1, we can interpret rings geometrically by means of their spectra, and likewise well-behaved sheaves of algebras on schemes or complex-analytic spaces by means of relative or analytic spectra. In general, we should seek geometric intuition for algebraic objects by similarly associating spaces to them in some natural contravariant way, such that we think of their elements as functions of some kind on the associated space.

Example 2.2.1. Consider a finite-dimensional vector space $M \cong \mathbb{C}^n$ over $\mathbb{C}$, regarded as a finitely-generated module. Then we can think of elements of M as linear functions on its dual $M^\vee$. To realize $M^\vee$ geometrically, we use the **symmetric algebra**

$$\operatorname{Sym}(M) := \bigoplus_{k=0}^{\infty} M^{\otimes k} \Big/ \big\langle a \otimes b - b \otimes a \mid a, b \in M \big\rangle,$$

where $M^{\otimes 0}$ is taken to be $\mathbb{C}$.

Concretely, if $v_1, \ldots, v_n$ form a basis for M, we have $\operatorname{Sym}(M) \cong \mathbb{C}[v_1, \ldots, v_n]$. We can thus recover $M^\vee$ as a scheme by taking $M^\vee = \operatorname{Spec}\operatorname{Sym}(M)$, and as a complex-analytic space by then taking the analytification.

More generally, if we have a locally free sheaf $\mathcal{E}$ of rank n on a scheme or complex-analytic space X, we can build an associated vector bundle of rank n over X by taking $\underline{\operatorname{Spec}} \operatorname{Sym}(\mathcal{E})$ or $\operatorname{Specan} \operatorname{Sym}(\mathcal{E})$ respectively, where $\mathcal{E}$ is recoverable as the sheaf of linear forms on the vector bundle — as in the vector space case, we think of this as a geometric realization of the dual object. In all of these cases, the operations are functorial, and in fact we can phrase our observations in terms of an anti-equivalence of categories — before we go into detail, however, it will be useful to introduce the more general version of this construction for arbitrary coherent sheaves. We will revisit vector bundles after we have done so, in Subsection 2.2.3, before moving on to the further generality provided by cones in Subsection 2.2.4.

2.2.1 Linear Fiber Spaces in the $\mathbb{C}$-Analytic Setting

To geometrically interpret coherent sheaves, we will need a notion more general than that of a vector bundle. We introduce it here, following the exposition in [Gro61b; Fis76].

Definition 2.2.2 ([Gro61b])**.** Let S be a complex-analytic space. Then the category of **linear fiber spaces** over S is defined to be the category of unitary $S \times \mathbb{C}$-modules in the category of complex-analytic spaces over S. Concretely, this means that a linear fiber space is given by a complex-analytic space L over S together with maps $L \times_S L \xrightarrow{+} L$ and $(S \times \mathbb{C}) \times_S L \xrightarrow{\cdot} L$ over S and a section $0 : S \to L$ of the projection $L \to S$ such that the diagrams corresponding to the module axioms for the operations $+$ and $\cdot$ commute, where $S \times \mathbb{C}$ is a ring in the category of complex-analytic spaces over S with the usual operations of fiberwise complex addition and multiplication. Likewise, a map of linear fiber spaces is a map $L \to L'$ of complex-analytic spaces over S which commutes with both operations on each space, as given by the commutativity of the appropriate diagrams.

Intuitively, a linear fiber space can be thought of as a generalization of a vector bundle such that the fibers are allowed to have differing dimensions from point to point. Of course, the possible failure of the spaces involved to be reduced complicates this picture slightly, and should be taken into account when using it for intuition.

As previously alluded to, there is a way to associate to any coherent sheaf on a complex-analytic space a corresponding linear fiber space:

Proposition 2.2.3. *Let S be a complex-analytic space and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_S$-modules. Then* $\operatorname{Specan}\operatorname{Sym}(\mathcal{F})$ *is a linear fiber space over S when endowed with the operations induced by the maps $\mathcal{F} \to \mathcal{F} \oplus \mathcal{F}$ and* $\operatorname{Sym}(\mathcal{F}) \to \operatorname{Sym}(\mathcal{F})[t]$ *given on sections of $\mathcal{F}$ as $x \mapsto x \oplus x$ and $x \mapsto xt$ respectively, and this construction is contravariantly functorial.*

Proof (or see [AM86]) The linear fiber space corresponding to a coherent sheaf was constructed in [Gro61b], while the analytic spectrum of a sheaf of algebras was constructed in [Hou61]. For a given coherent sheaf $\mathcal{F}$, we can verify that the construction of [Gro61b] agrees with the analytic spectrum of the symmetric algebra either by direct inspection of both constructions or by noting that the former represents the functor $(T \xrightarrow{f} S) \mapsto \operatorname{Hom}_{\mathcal{O}_T\text{-mod}}(f^*\mathcal{F}, \mathcal{O}_T)$ from complex-analytic spaces over S to sets and the latter the functor $(T \xrightarrow{f} S) \mapsto \operatorname{Hom}_{\mathcal{O}_T\text{-alg}}(f^* \operatorname{Sym}_{\mathcal{O}_S}(\mathcal{F}), \mathcal{O}_T)$; these are naturally isomorphic since $\operatorname{Sym}(-)$ commutes with f^* (e.g., Proposition A2.2 of [Eis04]) and is the left adjoint to the forgetful functor from sheaves of algebras to sheaves of modules. That the operations are as claimed can likewise be verified from the definitions, and functoriality follows by using the functoriality of $\operatorname{Specan}(-)$ and $\operatorname{Sym}(-)$ and verifying directly that the resulting maps commute with the operations.

Our general philosophy suggests we should be able to retrieve the coherent sheaf by considering functions of some kind on the resulting space, and since we are working with modules we should expect these to be linear. We formalize this as follows.

Definition 2.2.4 (e.g., [Fis76])**.** Let S be a complex-analytic space and L a linear fiber space over S. For any open subspace U of S, we set $L|_U := L \times_S U$, a linear fiber space over U, and let $\operatorname{Hom}_U(L|_U, U \times \mathbb{C})$ denote the set of homomorphisms of linear fiber spaces over U as defined in Definition 2.2.2. Then we denote the sheaf of $\mathcal{O}_S$-

modules given by $U \mapsto \mathrm{Hom}_U(L|_U, U \times \mathbb{C})$ by $\mathcal{L}_S(L)$ and call it the **sheaf of linear forms on** L.

It can be seen straightforwardly that this construction is functorial. That the resulting sheaf is coherent and that it gives the result we expect when we apply it to the analytic spectrum of the symmetric algebra are encapsulated in the following result, called the **Fischer-Prill Theorem**, which identifies the study of linear fiber spaces with that of coherent sheaves.

Theorem 2.2.5 (e.g., [Fis76; AM86])**.** *Let S be a complex-analytic space. The contravariant functors* $\operatorname{Specan}\operatorname{Sym}(-)$ *and* $\mathcal{L}_S(-)$ *define an anti-equivalence between the category of coherent $\mathcal{O}_S$-modules and the category of linear fiber spaces over S.*

We will not reproduce the proof, the sketch of which can be found in [Fis76]. A complete proof based on the results we will review in Subsection 2.2.4 can be found in [AM86].

The Fischer-Prill Theorem gives us exactly what we hoped for at the outset — a rigorous way of regarding the algebraic objects under consideration, coherent sheaves on a complex-analytic space, as collections of functions on associated geometric spaces, in the fashion of rings and affine schemes.

2.2.2 Linear Fiber Spaces Over Schemes

We now wish to do the same for coherent sheaves in the algebraic context — say, on a Noetherian scheme. It is not too difficult to formulate a definition of linear fiber spaces corresponding to the complex-analytic case:

Definition 2.2.6 (cf. Section 1.7 of [Gro61a])**.** Let S be a Noetherian scheme. Then the category of **linear fiber spaces** over S is defined to be the category of unitary $\mathbb{A}^1_S$-modules in the category of schemes over S. Concretely, this means that a linear fiber space is given by a scheme L over S together with maps $L \times_S L \xrightarrow{+} L$ and $\mathbb{A}^1_S \times_S L \xrightarrow{\cdot} L$ over S and a section $0 : S \to L$ of the projection $L \to S$ such that the diagrams corresponding to the module axioms for the operations $+$ and $\cdot$ commute, where $\mathbb{A}^1_S$ is a ring in the category of schemes over S with the operations induced by the usual addition and multiplication on $\mathbb{A}^1_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[t]$. Likewise, a map of linear fiber spaces is a map $L \to L'$ of schemes over S which commutes with both operations on each space, as given by the commutativity of the appropriate diagrams.

However, some care must be taken — as we have already seen in Subsection 2.2.1, every linear fiber space in the complex-analytic setting arises from a coherent sheaf, but, since schemes are much more versatile objects, this is no longer true in the algebraic setting without additional restriction. As an easy example, we can take a linear fiber space which is too large to arise from a coherent sheaf:

Example 2.2.7. Let $R = \mathbb{C}[x_1, x_2, x_3, \ldots]$ be the polynomial ring in countably many variables. Then $\operatorname{Spec} R$ is a linear fiber space over $\operatorname{Spec} \mathbb{C}$ with the operations given by $x_i \mapsto x_i \otimes 1 + 1 \otimes x_i$ and $x_i \mapsto tx_i$. However, it cannot arise as $\underline{\operatorname{Spec}} \operatorname{Sym}(\mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on $\operatorname{Spec} \mathbb{C}$ — that is, any finite-dimensional $\mathbb{C}$-vector space — since it is not of finite type over $\operatorname{Spec} \mathbb{C}$.

To resolve this, one could restrict the definition of linear fiber spaces to require that they be of finite type over the base, or work with quasicoherent sheaves, not just coherent ones. While results along the lines of the Fischer-Prill Theorem 2.2.5 may then be true, we will not pursue them here — since we are mainly interested in being able to geometrically visualize coherent sheaves, it is enough for our purposes to show an anti-equivalence between the category of coherent sheaves and the full subcategory of the category of linear fiber spaces which is the image of the functor $\underline{\operatorname{Spec}} \operatorname{Sym}(-)$. We begin by verifying that this statement makes sense:

Proposition 2.2.8. *Let S be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_S$-modules. Then $\underline{\operatorname{Spec}} \operatorname{Sym}(\mathcal{F})$ is a linear fiber space over S when endowed with the operations induced by the maps $\mathcal{F} \to \mathcal{F} \oplus \mathcal{F}$ and $\operatorname{Sym}(\mathcal{F}) \to \operatorname{Sym}(\mathcal{F})[t]$ given on sections of $\mathcal{F}$ as $x \mapsto x \oplus x$ and $x \mapsto xt$ respectively, and this construction is contravariantly functorial.*

Proof We observe that the claims we must verify and the objects under consideration are all local on S, so we can reduce to the case where $S = \operatorname{Spec} R$ is an affine scheme, for R a Noetherian ring. In this situation, $\mathcal{F}$ will be given by a finitely-generated R-module M (e.g., Proposition II.5.4 of [Har77]), so it suffices to show that the claimed operations make $\operatorname{Sym}(M)^{\mathrm{op}}$ a unitary $R[t]^{\mathrm{op}}$-module in the opposite category of finitely-generated R-algebras and that the construction is functorial; both claims can be verified directly by diagram chases.

We can likewise define a notion of linear forms, exactly as in the complex-analytic setting:

Definition 2.2.9. Let S be a Noetherian scheme and L a linear fiber space over S. For any open subspace U of S, we set $L|_U := L \times_S U$, a linear fiber space over U, and let $\operatorname{Hom}_U(L|_U, \mathbb{A}^1_U)$ denote the set of homomorphisms of linear fiber spaces over U. Then we denote the sheaf of $\mathcal{O}_S$-modules given by $U \mapsto \operatorname{Hom}_U(L|_U, \mathbb{A}^1_U)$ by $\mathcal{L}_S(L)$ and call it the **sheaf of linear forms on** L.

As in the analytic case, the functoriality of $\mathcal{L}_S(-)$ is immediate. We are now in a position to show that coherent sheaves correspond to linear fiber spaces of a certain kind:

Theorem 2.2.10. *Let S be a Noetherian scheme. The contravariant functors $\underline{\operatorname{Spec}}\operatorname{Sym}(-)$ and $\mathcal{L}_S(-)$ define an anti-equivalence between the category of coherent $\mathcal{O}_S$-modules and a full subcategory of the category of linear fiber spaces over S.*

Proof As in the proof of Proposition 2.2.8, we can verify our claims locally, so we reduce to the case where $S = \operatorname{Spec} R$ for R a Noetherian ring.

We first verify that $\mathcal{L}_S(\underline{\operatorname{Spec}}\operatorname{Sym}(-))$ is naturally isomorphic to the identity functor on the category of coherent sheaves of $\mathcal{O}_S$-modules. Let M be a finitely-generated R-module. Then $\Gamma(S, \mathcal{L}_S(\underline{\operatorname{Spec}}\operatorname{Sym}(M)))$ is the subset of $\operatorname{Hom}_{R\text{-alg}}(R[t], \operatorname{Sym}(M))$ consisting of those ring maps such that the corresponding maps of affine schemes satisfy the linear fiber space axioms, with the R-module structure such that $r \cdot \phi$

is defined as the precomposition of ϕ with the R-algebra map $R[t] \to R[t]$ given by $t \mapsto rt$. Observe then that $\mathrm{Hom}_{R\text{-alg}}(R[t], \mathrm{Sym}(M)) \cong \mathrm{Sym}(M)$ as R-modules — to show that $\Gamma(S, \mathcal{L}_S(\underline{\mathrm{Spec}}\,\mathrm{Sym}(M))) \cong M$, we thus need only verify that satisfying the linear fiber space axioms is equivalent to being homogeneous of degree 1 under the natural grading of $\mathrm{Sym}(M)$.

For an R-algebra map ϕ to satisfy these axioms, it must in particular commute with scalar multiplication — that is, the following diagram must commute:

$$\begin{array}{ccc} R[t] & \xrightarrow{\quad\phi\quad} & \mathrm{Sym}(M) \\ \downarrow & & \downarrow{\scriptstyle\mu} \\ R[\tau, t] & \xrightarrow{\mathrm{id}\otimes\phi} & R[\tau] \otimes_R \mathrm{Sym}(M) \end{array}$$

Here the vertical arrows are the ring maps corresponding to the multiplication maps $\mathbb{A}^1_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S$ and $\mathbb{A}^1_S \times_S \underline{\mathrm{Spec}}\,\mathrm{Sym}(M) \to \underline{\mathrm{Spec}}\,\mathrm{Sym}(M)$. Since the left vertical map is given by $t \mapsto \tau t$, this is to say that the element $f \in \mathrm{Sym}(M)$ corresponding to ϕ satisfies $\mu(f) = \tau f$. Letting $f = \sum_{d=0}^{n} f_d$ be the decomposition of f into homogeneous parts under the grading of $\mathrm{Sym}(M)$, we can see from the definition that $\mu(f) = \sum_{d=0}^{n} \tau^d f_d$; taking this equality in each degree individually and observing that $\tau^d - \tau$ is always a non-zerodivisor in $R[\tau] \otimes_R \mathrm{Sym}(M) = \mathrm{Sym}(M)[\tau]$ for $d \neq 1$ demonstrates that $f_d = 0$ for all $d \neq 1$, as desired. Hence every map of linear fiber spaces corresponds to homogeneous degree-1 element of $\mathrm{Sym}(M)$; it is likewise easy to verify for any such element that the corresponding map will be a morphism of linear fiber spaces.

As such, $\Gamma(S, \mathcal{L}_S(\underline{\mathrm{Spec}}\,\mathrm{Sym}(M))) \cong M$, and it is straightforward to verify that the isomorphism is natural. Since our reasoning can be applied to any distinguished open subset of S by changing the choice of R, we can see that $\mathcal{L}_S(\underline{\mathrm{Spec}}\,\mathrm{Sym}(M))$ is in fact naturally isomorphic to M as sheaves on $S = \mathrm{Spec}\, R$, as desired. Hence the category of coherent $\mathcal{O}_S$-modules is anti-equivalent to the subcategory of the category of linear fiber spaces which is the image of $\underline{\mathrm{Spec}}\,\mathrm{Sym}(-)$, and it remains only to show that this is a full subcategory by verifying that this functor is surjective on Hom-sets.

To do so, we observe that, for R-modules M and N, the morphisms $\underline{\operatorname{Spec}} \operatorname{Sym}(M) \to \underline{\operatorname{Spec}} \operatorname{Sym}(N)$ as S-schemes are precisely the R-algebra maps $\operatorname{Sym}(N) \to \operatorname{Sym}(M)$, and by the adjunction with the forgetful functor these correspond to R-module homomorphisms $N \to \operatorname{Sym}(M)$. A slight generalization of our reasoning above in the case $N = R$ (and hence $\operatorname{Sym}(N) \cong R[t]$) then shows that those homomorphisms corresponding to maps of linear fiber spaces are precisely those which have image contained in the homogeneous degree-1 part of $\operatorname{Sym}(M)$ — that is, M. It can thus be seen that every map $\underline{\operatorname{Spec}} \operatorname{Sym}(M) \to \underline{\operatorname{Spec}} \operatorname{Sym}(N)$ of linear fiber spaces arises from the corresponding map $N \to M$ of R-modules, completing the result.

Hence we again have a realization of our algebraic objects in terms of functions on a particular kind of geometric object.

2.2.3 Vector Bundles

In particular, we can now return to vector bundles and provide a link between algebraic and geometric viewpoints — compare Exercise II.5.18 of [Har77].

Proposition 2.2.11. *Vector bundles of rank n and locally free sheaves of rank n correspond under the equivalences of Theorems 2.2.5 and 2.2.10 in the analytic and algebraic settings.*

This is essentially because we can use the isomorphisms $S \times \mathbb{C}^n \cong \operatorname{Specan} \operatorname{Sym}(\mathcal{O}_S^{\oplus n})$ and $\mathbb{A}_S^n \cong \underline{\operatorname{Spec}} \operatorname{Sym}(\mathcal{O}_S^{\oplus n})$ for S a complex-analytic space and Noetherian scheme respectively to show the equivalence of the local trivialization requirements for each type of object.

For our purposes, there are two linear fiber spaces for which the condition of being a vector bundle will be especially significant: those corresponding to the **tangent** and **cotangent sheaves**.

Definition 2.2.12 (e.g., [Har77; Tei77; Vak23])**.** Let $\phi : X \to Y$ be a map of schemes. Let $\Omega_{X/Y}$ be the quasicoherent sheaf of $\mathcal{O}_X$-modules representing the covariant functor which takes each quasicoherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules to the set of $\phi^{-1}\mathcal{O}_Y$-algebra sheaf maps $d : \mathcal{O}_X \to \mathcal{F}$ satisfying the **Leibniz rule** $d(ab) = ad(b) + bd(a)$ — such maps are called **derivations**. Then $\Omega_{X/Y}$, which we also denote by Ω_ϕ, is called the **sheaf of relative Kähler differentials**, or **relative cotangent sheaf**, of X over Y. We call the corresponding space $\underline{\operatorname{Spec}} \operatorname{Sym}(\Omega_{X/Y})$ the **relative tangent space** of X over Y.

Similarly, we define $\Theta_{X/Y}$ or Θ_ϕ to be the dual sheaf $\mathscr{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ of $\Omega_{X/Y}$ and call it the **sheaf of derivations**, or **relative tangent sheaf**, of X over Y. We call the corresponding space $\underline{\operatorname{Spec}} \operatorname{Sym}(\Theta_{X/Y})$ the **relative cotangent space** of X over Y.

The definitions for a map of complex-analytic spaces are similar — however, for technical reasons, we must define $\Omega_{X/Y}$ as the representative of the functor described only on the category of quasicoherent $\mathcal{O}_X$-modules $\mathcal{F}$ which are moreover **separated** — that is, whose stalks $M = \mathcal{F}_x$ for $x \in X$ satisfy $\bigcap_{k=0}^{\infty} \mathfrak{m}^k M = 0$ for $\mathfrak{m}$ the maximal ideal in the local ring.

If X is a scheme over a perfect field, such as $\mathbb{C}$, and ϕ is the structure morphism, or if X is a complex-analytic space and Y is a reduced point, we drop the word "relative" from the terminology and suppress Y in the notation. In this case, we denote the tangent space by TX and the cotangent space by T^*X.

In Proposition 2.3.16, we will introduce an alternate definition which makes it easy to verify that the cotangent and tangent sheaves are indeed coherent so long as, in the scheme case, the schemes involved are locally Noetherian. As mentioned, we will be particularly interested in the case where these sheaves are locally free:

Definition 2.2.13 (e.g., [Har77; Vak23])**.** Let X be a connected finite-type scheme over a perfect field or a connected complex-analytic space. If TX is a vector bundle with rank equal to $\dim X$, we say that X is **smooth**. If X is not connected, we say that it is smooth if and only if all its components are. In this case we call TX and T^*X respectively the **tangent** and **cotangent bundles** of X.

We say that a morphism $X \to Y$ of schemes is **smooth of relative dimension** n if it is locally finitely presented and flat of relative dimension n and $\Omega_{X/Y}$ is a vector bundle of rank n — the definition in the complex-analytic case omits the finite presentation requirement but is otherwise the same.

Smoothness will be important in Chapter 3, where we discuss the theory of stratifications. For now, we content ourselves with the following **generic smoothness** result for algebraic varieties:

Theorem 2.2.14 (e.g., [Vak23])**.** *Let X be an integral finite-type scheme over a perfect field. Then there exists a dense open subscheme $U \subseteq X$ such that U is smooth.*

It is not difficult to see that this result extends to reduced finite-type schemes over perfect fields which are not necessarily irreducible.

2.2.4 Cones

For the remainder of this chapter our primary interest will be in objects more general than linear fiber spaces — cones — which retain the multiplication by scalars but need not have any additive structure. As we will see in the cases of Section 2.3's normal cone and Section 3.2's relative conormal space, these objects play an important role in a variety of different settings within algebraic and complex-analytic geometry.

We first give the definition and basic facts in the complex-analytic case, as laid out in [AM86].

Definition 2.2.15. Let S be a complex-analytic space. Then the category of **cones** over S is defined to be the category of modules over the monoid-with-zero $S \times \mathbb{C}$ in the category of complex-analytic spaces over S. Concretely, this means that a cone is given by a complex-analytic space C over S together with a map $\mathbb{C} \times C \cong (S \times \mathbb{C}) \times_S C \xrightarrow{\cdot} C$ over S and a section $0 : S \to C$ such that, when $S \times \mathbb{C}$ is given the usual fiberwise complex multiplication (also denoted $\cdot$), the following diagrams commute:

$$
\begin{array}{ccc}
\mathbb{C}\times\mathbb{C}\times C & \xrightarrow{\cdot\times \mathrm{id}_C} & \mathbb{C}\times C \\
{\scriptstyle \mathrm{id}_{\mathbb{C}}\times\cdot}\downarrow & & \downarrow{\scriptstyle \cdot} \\
\mathbb{C}\times C & \xrightarrow{\ \cdot\ } & C
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{\mathrm{id}_C} & C \\
{\scriptstyle 1\times\mathrm{id}_C}\downarrow & \nearrow{\scriptstyle \cdot} & \\
\mathbb{C}\times C & &
\end{array}
\qquad
\begin{array}{ccc}
C & \longrightarrow & S \\
{\scriptstyle 0\times\mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle 0} \\
\mathbb{C}\times C & \xrightarrow{\ \cdot\ } & C
\end{array}
$$

Likewise, a map of cones is a map $\phi : C \to C'$ of complex-analytic spaces over S such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C}\times C & \xrightarrow{\ \cdot\ } & C \\
{\scriptstyle \mathrm{id}_{\mathbb{C}}\times\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\
\mathbb{C}\times C' & \xrightarrow{\ \cdot\ } & C'
\end{array}
$$

Intuitively, this is to say that a cone over S is a space over S such that each fiber has a scaling action of $\mathbb{C}$ by multiplication, although as usual the existence of nilpotents means that the phrasing in terms of fibers is not literally correct.

Per our running theme, there is an anti-equivalence between complex-analytic cones and a certain kind of algebraic object:

Theorem 2.2.16 ([AM86])**.** *Let S be a complex-analytic space. Then the contravariant functor* Specan$(-)$ *gives an anti-equivalence of categories between the category of finitely-presented quasicoherent sheaves of* $\mathbb{N}$*-graded* $\mathcal{O}_S$*-algebras with degree-zero part* $\mathcal{O}_S$ *and the category of complex-analytic cones over S.*

The scalar multiplication on $\operatorname{Specan}\mathcal{A}$, for $\mathcal{A}$ such a sheaf of algebras, is induced by the map $\mathcal{A} \to \mathcal{A}[t]$ which takes each homogeneous degree-d element a to $t^d a$. The sheaf of homogeneous algebras corresponding to a cone $C \xrightarrow{\pi} S$ is recovered by taking, in each degree d, the subsheaf of $\pi_*\mathcal{O}_C$ consisting of those holomorphic functions on which the $\mathbb{C}$-multiplication acts by dth powers — see [AM86] for details. Note that the requirement on the degree-zero part serves to ensure that the analytic spectrum is genuinely a cone over S — if we omit it, taking the analytic spectrum instead gives a cone over $\operatorname{Specan}\mathcal{A}_0$.

If a sheaf of algebras $\mathcal{A}$ of the sort we are considering is generated in degree 1, we find in particular that $\operatorname{Specan}\mathcal{A}$ is a closed subspace of the linear fiber space $\operatorname{Specan}\operatorname{Sym}(\mathcal{A}_1)$

— the grading on $\mathcal{A}$ tells us exactly that this subspace is invariant under the $\mathbb{C}$-action, so in this case we can intuitively think of cones as subspaces of linear fiber spaces which restrict in each fiber to unions of lines through the origin, with the usual caveats about non-reduced structure. More generally, we have the following analogue to Definition 2.1.10:

Definition 2.2.17 ([AM86])**.** Let S be a complex-analytic space and $\mathcal{A} = \bigoplus_{d=0}^{\infty} \mathcal{A}_d$ a finitely-presented sheaf of $\mathbb{N}$-graded algebras on S. Then the unique complex-analytic space over $\operatorname{Specan} \mathcal{A}_0$ which is the quotient of the complement of the zero section in $\operatorname{Specan} \mathcal{A}$ by the natural $\mathbb{C}^*$-action is called the **analytic homogeneous spectrum of $\mathcal{A}$** and denoted by $\operatorname{Projan} \mathcal{A}$. If C is a complex-analytic cone, the analytic homogeneous spectrum of the corresponding sheaf of algebras is called the **projectivization of C** and denoted by $\mathbb{P}C$.

The construction, the details of which can be found in [AM86], is carried out by charts exactly as one would expect from that of the algebraic relative homogeneous spectrum discussed in Subsection 2.1.3. Hence we can think of complex-analytic cones over S as being the affine cones over projective morphisms to S — if $\mathcal{A}$ is generated in degree 1, then every local choice of $n+1$ such generators gives a local embedding of $\operatorname{Projan} \mathcal{A}$ into $S \times \mathbb{P}^n$, and more generally a local choice of homogeneous generators not necessarily in degree 1 gives a local embedding into some *weighted* projective space over S.

If a cone does arise from an algebra generated in degree 1, it is useful to note that, for the purpose of checking flatness, we can work with either a complex-analytic cone or the corresponding sheaf of algebras. To prove this, we need the following lemma showing that flatness is an open condition, analogously to Theorem 24.3 of [Mat89] and Theorem IV.9 of [Fri67]:

Lemma 2.2.18. *Let $\phi : X \to Y$ be a map of complex-analytic spaces and $\mathcal{A}$ a finitely-presented quasicoherent sheaf of $\mathcal{O}_X$-algebras which can be written as a (potentially infinite) direct sum of coherent $\mathcal{O}_X$-modules. Then, for each point $x \in X$ and $y := \phi(x)$, the set of points in $\operatorname{Spec} \mathcal{A}_x$ where $\mathcal{A}_x$ is flat over $\mathcal{O}_{Y,y}$ is open.*

Proof The proof is by the **topological Nagata criterion**, for which see Theorem 24.2 of [Mat89]. It proceeds essentially as does the proof of Theorem 24.3 of [Mat89], but there are two complications to address.

The first is that this proof requires $\mathcal{A}_x$ to be $\mathfrak{p}$-adically ideal-separated for certain prime ideals $\mathfrak{p}$ of $\mathcal{O}_{Y,y}$ — this can be shown using the hypothesis that $\mathcal{A}$ decomposes as a direct sum of coherent modules, since then $\mathcal{A}_x$ is a direct sum of finitely-generated $\mathcal{O}_{X,x}$-modules. We can verify the separation of each individually, using the corresponding ideal $\mathfrak{p}\mathcal{O}_{X,x}$; since this is contained in the maximal ideal, the separation now follows from the Krull Intersection Theorem (e.g., Corollary 5.4 of [Eis04]).

The second is that the generic freeness theorem used in [Mat89] does not apply for $\mathcal{A}_x$ over $\mathcal{O}_{Y,y}$. Since X is locally a closed subspace of $\mathbb{C}^n$ for some n, the required analogue is essentially Theorem II.1 of [Fri67], with the important difference that we now use our finite-type sheaf $\mathcal{A}$ of algebras instead of a coherent sheaf. This does not introduce any complications, however, other than requiring us to apply the algebraic generic flatness lemma slightly differently, using the fact that the associated graded of $\mathcal{A}$ with respect to the chosen ideal is itself a finite-type algebra; once we have thus guaranteed a consistent choice of localization, the proof can proceed unaltered on each coherent summand individually.

We can now prove the claimed result:

Proposition 2.2.19. *Let $\phi : X \to Y$ be a map of complex-analytic spaces, C a complex-analytic cone over X, and $\mathcal{A}$ the corresponding sheaf of algebras on X. Suppose that $\mathcal{A}$ is generated in degree 1. Then C is flat over Y if and only if $\mathcal{A}$ is flat over $\phi^{-1}\mathcal{O}_Y$.*

Proof Since flatness is a local property, we can suppose without loss of generality that $\mathcal{A} = \mathcal{O}_X[t_0, \ldots, t_n]/(f_1, \ldots, f_r)$ for functions $f_1, \ldots, f_r$ homogeneous in the t_i. Hence C is the closed subspace of $X \times \mathbb{C}^{n+1}$ cut out by these functions.

Consider a point $c \in C$, let $x \in X$ be the point in X it is mapped to, and set

$y := \phi(x)$. Let $R := \mathcal{O}_{Y,y}$, $S := \mathcal{O}_{X,x}$, and $T := \mathcal{O}_{C,c}$ be the local rings and $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{o}$ be their respective maximal ideals. Observe that the stalk $(\phi^{-1}\mathcal{O}_Y)_x = R$ and, if we let $c_0, \ldots, c_n$ be the $\mathbb{C}^{n+1}$-coordinates of c, we have $T = (S \hat{\otimes} \mathbb{C}\{t_0 - c_0, \ldots, t_n - c_n\})/(f_1, \ldots, f_r)$, where $\hat{\otimes}$ denotes the **analytic tensor product** of convergent power series rings (e.g., [Fis76]). If we let $\tilde{T}$ denote the localization of the stalk $\mathcal{A}_x = S[t_0, \ldots, t_n]/(f_1, \ldots, f_r)$ at the maximal ideal $\mathfrak{n} + (t_0 - c_0, \ldots, t_n - c_n)$, Proposition 2.1.6 and the preservation of faithful flatness under pullback imply that the natural map $\tilde{T} \to T$ is faithfully flat.

Suppose that $\mathcal{A}$ is flat over $\phi^{-1}\mathcal{O}_Y$. Then, for each such c, we have that $\mathcal{A}_x$ is flat over R and hence $\tilde{T}$ is as well by the flatness of localization; since $\tilde{T} \to T$ is flat, it follows that T is flat over R. Since c was arbitrary, we find that C is flat over Y, as desired.

On the other hand, suppose that $\mathcal{A}$ is not flat over $\phi^{-1}\mathcal{O}_Y$. Let x be a point where, when $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are as before, $\hat{T} := \mathcal{A}_x$ is not flat over R. Then, by Lemma 2.2.18, which we can apply since $\mathcal{A}$ is the direct sum of its (coherent) graded pieces, the non-flat locus of $\operatorname{Spec} \hat{T}$ over $\operatorname{Spec} R$ is closed in $\operatorname{Spec} \hat{T}$; we claim that it contains the point corresponding to $\mathfrak{n} + (t_0, \ldots, t_n)$. By hypothesis, the non-flat locus contains at least one point p; let $\mathfrak{p}$ be the prime ideal corresponding to p and suppose first that $\mathfrak{p} \not\supseteq (t_0, \ldots, t_n)$.

Then, without loss of generality, $t_0 \notin \mathfrak{p}$, so p is contained in $\operatorname{Spec} \hat{T}_{t_0}$. However, it is not difficult to see that the natural map $(\hat{T}_{t_0})_0[t^{\pm 1}] \to \hat{T}_{t_0}$ taking t to t_0, where $(\hat{T}_{t_0})_0$ is the degree-zero part, is an isomorphism with inverse given by $t_i \mapsto t \cdot t_i/t_0$. Hence $\operatorname{Spec} \hat{T}_{t_0}$ is the trivial punctured line bundle over $\operatorname{Spec}(\hat{T}_{t_0})_0$, and thus faithfully flat over this space as well. Setting $\mathfrak{p}' = \mathfrak{p}\hat{T}_{t_0} \cap (\hat{T}_{t_0})_0$, we then find that $\operatorname{Spec}(\hat{T}_{t_0})_0$ is not flat over R at the point corresponding to $\mathfrak{p}'$ by our hypothesis on p and the flatness of the bundle map, and so $\operatorname{Spec} \hat{T} \supset \operatorname{Spec} \hat{T}_{t_0}$ is not flat over R at the generic point of the preimage of the vanishing of $\mathfrak{p}'$ by the faithfulness result. Let $\mathfrak{p}'' = \mathfrak{p}'\hat{T}_{t_0}$ be the prime ideal of $\hat{T}$ corresponding to this point; it is clear from the construction that $\mathfrak{p}''$ is homogeneous.

Thus, by taking $\mathfrak{p}''$ instead, we can suppose without loss of generality that $\mathfrak{p}$ is homogeneous. In particular, $\mathfrak{p} + (t_0, \dots, t_n)$ cannot be the unit ideal since $\mathfrak{p}$ itself is not, so the closure of p meets the vanishing of $(t_0, \dots, t_n)$. Since the non-flat locus is closed, it follows that there are points of this vanishing at which $\operatorname{Spec} \hat{T}$ is not flat. On the other hand, if our original supposition does not hold and $\mathfrak{p} \supseteq (t_0, \dots, t_n)$, this is obviously true as well. However, since $\hat{T}/(t_0, \dots, t_n) \cong S$ and every closed subset of $\operatorname{Spec} S$ contains the point corresponding to $\mathfrak{n} + (t_0, \dots, t_n)$, we then see that the non-flat locus contains this point as desired.

Denoting this point by c, we then see with our usual notations that the localization $\tilde{T}$ of $\hat{T}$ at c is not flat over R. By the faithful flatness of $\tilde{T} \to T = \mathcal{O}_{C,c}$ given in Proposition 2.1.6, we can see that the complex-analytic cone C is not flat over Y at c. This proves the result.

2.3 Tangent and Normal Cones

We are now ready to introduce the titular object of this chapter, the normal cone. With the machinery of Subsection 2.2.4, the definition is easy to state:

Definition 2.3.1. Let $i : Y \hookrightarrow X$ be a closed inclusion of schemes or complex-analytic spaces, with $\mathcal{I}$ the sheaf of ideals on X cutting out Y. Then the **associated graded sheaf of $\mathcal{I}$ in $\mathcal{O}_X$** is the sheaf of $\mathcal{O}_X$-algebras given by

$$\operatorname{gr}_{\mathcal{I}} \mathcal{O}_X := \bigoplus_{k=0}^{\infty} \mathcal{I}^k / \mathcal{I}^{k+1}$$

with the product structure induced by the natural multiplication maps $\mathcal{I}^p \otimes_{\mathcal{O}_X} \mathcal{I}^q \to \mathcal{I}^{p+q}$. The cone over Y corresponding to $i^* \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$ (which is in fact equal to $i^{-1} \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$, since the associated graded is already an $\mathcal{O}_X/\mathcal{I} = i_* \mathcal{O}_Y$-module) is called the **normal cone to Y in X** and denoted by $C_Y X$.

If f is a section of $\mathcal{O}_X$ over an open set U not contained in $\bigcap_{k=0}^{\infty} \mathcal{I}^k(U)$, we call its equivalence class in $(\mathcal{I}^k/\mathcal{I}^{k+1})(U) \subseteq (\operatorname{gr}_{\mathcal{I}} \mathcal{O}_X)(U)$, for k largest such that $f \in \mathcal{I}^k$, its **initial form**, and denote it by $\operatorname{in}_{\mathcal{I}(U)} f$. If it does not fulfill this condition, we set $\operatorname{in}_{\mathcal{I}(U)} f := 0$.

Note that the definition of the normal cone in the complex-analytic case relies on the associated graded sheaf being finitely presented — this is a consequence of the Oka Coherence Theorem 2.1.1.

We will discuss the geometric intuition in detail momentarily in Subsections 2.3.1 and 2.3.2. In doing so, we will make use of the following construction, which realizes the normal cone as a deformation of the ambient space.

Definition 2.3.2 (e.g., [Eis04])**.** Let $i : Y \hookrightarrow X$ be a closed inclusion of schemes or complex-analytic spaces, with $\mathcal{I}$ the sheaf of ideals on X cutting out Y. Then the **Rees algebra sheaf of $\mathcal{O}_X$ with respect to $\mathcal{I}$** is the subsheaf of the sheaf $\mathcal{O}_X[t^{\pm 1}]$ of $\mathcal{O}_X$-algebras given by

$$\mathcal{R}(\mathcal{O}_X, \mathcal{I}) := \mathcal{O}_X[t, t^{-1}\mathcal{I}] = \mathcal{O}_X[t] + \sum_{k=1}^{\infty} \mathcal{I}^k t^{-k} \subseteq \mathcal{O}_X[t^{\pm 1}].$$

If X is a scheme over a field $\mathbb{F}$, we call the natural map $\phi : \underline{\operatorname{Spec}}\, \mathcal{R}(\mathcal{O}_X, \mathcal{I}) \to \mathbb{A}^1_{\mathbb{F}} = \operatorname{Spec} \mathbb{F}[t]$ the **deformation to the normal cone**, and similarly the map $\phi : \operatorname{Specan} \mathcal{R}(\mathcal{O}_X, \mathcal{I}) \xrightarrow{t} \mathbb{C}$ if X is a complex-analytic space.

As with the normal cone itself, the well-definedness in the complex-analytic case follows from the Oka Coherence Theorem 2.1.1.

The terminology "deformation to the normal cone" is justified by the following observations. First, we note that the fiber of ϕ over any nonzero $\mathbb{F}$-rational closed point of $\mathbb{A}^1_{\mathbb{F}}$ or, respectively, any nonzero point of $\mathbb{C}$ is simply X, since taking such a fiber corresponds to setting t equal to a nonzero, hence invertible, constant value. On the other hand, we can see by considering the quotient of $\mathcal{R}(\mathcal{O}_X, \mathcal{I})$ by t that the fiber over the origin in $\mathbb{A}^1_{\mathbb{F}}$ or $\mathbb{C}$ is precisely $C_Y X$. Thus ϕ realizes the normal cone as the special fiber of a map whose general fiber is the ambient space; to think of this as a deformation, we need only verify flatness, as discussed in Subsection 2.1.2.

Proposition 2.3.3 (e.g., [Eis04])**.** *Let $i : Y \hookrightarrow X$ be a closed inclusion of schemes over a field or complex-analytic spaces, with $\mathcal{I}$ the sheaf of ideals on X cutting out Y. Then the map ϕ of Definition 2.3.2 is flat.*

Proof The scheme case is immediate by working locally and using Corollary 6.11 of [Eis04]. In the complex-analytic case, we can see that ϕ defines the trivial X-bundle over $\mathbb{C}^*$, so we need only verify flatness at points over the origin in $\mathbb{C}$.

For a point $x \in X$, if we let $R := \mathcal{O}_{X,x}$, $\mathfrak{m}$ the maximal ideal of R, and $I = \mathcal{I}_x$, we have $\mathcal{R}(\mathcal{O}_X, \mathcal{I})_x = \mathcal{R}(R, I)$. This is a finitely-generated $R[t]$-algebra with generators $\tilde{g}_i := t^{-1} g_i$ for $g_1, \ldots, g_k$ generators of I; the points p of $\operatorname{Specan} \mathcal{R}(\mathcal{O}_X, \mathcal{I})$ lying over both x and the origin in $\mathbb{C}$ then correspond to ideals of the form $\mathfrak{p} = \mathfrak{m} + (t, \tilde{g}_1 - v_1, \ldots, \tilde{g}_k - v_k)$ for values $v_1, \ldots, v_k$ satisfying the relations. By Proposition 2.1.6 and the preservation of flatness under pullback, the map $\mathcal{R}(R, I)_{\mathfrak{p}} \to \mathcal{O}_{\operatorname{Specan} \mathcal{R}(\mathcal{O}_X, \mathcal{I}), p}$ is flat; since t is a non-zerodivisor in $\mathcal{R}(R, I)_{\mathfrak{p}}$ by the algebraic case of the theorem, it is then a non-zerodivisor in $\mathcal{O}_{\operatorname{Specan} \mathcal{R}(\mathcal{O}_X, \mathcal{I}), p}$ by flatness. Since $\mathbb{C}\{t\}$ is a principal ideal domain with maximal ideal generated by t, this proves the result by Corollary 6.3 of [Eis04].

Hence, as a first piece of geometric intuition about the normal cone, we can see that the ambient space X can be deformed to it — this gives, for instance, natural results on the dimensions of its fibers over points of Y (e.g., Theorem 10.10 of [Eis04]).

2.3.1 Intuition for the Tangent Cone

We now begin building more detailed geometric intuition by starting with the special case where Y is a single point in X. Consider the situation of a point x in a scheme or complex-analytic space X, letting $R = \mathcal{O}_{X,x}$ be the local ring of X at x and $\mathfrak{m}$ its maximal ideal.

As discussed in Subsection 2.1.2 following Definition 2.1.3, the first-order infinitesimal neighborhood $\operatorname{Spec} R/\mathfrak{m}^2$ carries information about the values and first derivatives of germs of functions on X at x. Hence, if we consider only those elements of $R/\mathfrak{m}^2$ which vanish at x — that is, elements of the maximal ideal $\mathfrak{m}/\mathfrak{m}^2$ — the information we obtain is precisely that of the first derivatives of such function germs alone. This motivates the following standard construction, which algebro-geometrically recovers and extends a familiar notion from differential geometry:

Definition 2.3.4 (e.g., [Har77])**.** Let X be a scheme (or complex-analytic space), $x \in X$ a point, $R = \mathcal{O}_{X,x}$, and $\mathfrak{m}$ the maximal ideal of R, with $\kappa = R/\mathfrak{m}$ the residue field. Then the **Zariski tangent space to X at x** is the κ-vector space $T_xX := \operatorname{Spec}\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2)$, where $\operatorname{Sym}(-)$ is taken over κ. For X locally Noetherian at x, if $\dim T_xX = \dim \operatorname{Spec} \mathcal{O}_{X,x}$, we say that x is a **regular point of X**; otherwise, when $\dim T_xX > \dim \operatorname{Spec} \mathcal{O}_{X,x}$, we say that it is a **singular point**.

In the complex-analytic setting, the regular points are precisely those around which the space is a manifold, with the Zariski tangent space recovering the usual one — note that here we are not considering non-reduced spaces to be manifolds. We introduced an analogue to the tangent bundle for a scheme over a perfect field already in Definition 2.2.12, and as one would expect the fiber of this space over each point will turn out to be the Zariski tangent space — we will discuss this, as well as the relationship between regularity and smoothness, in Subsection 3.1.1.

Throughout this subsection, we will illustrate the concepts under consideration using the example of the nodal cubic:

Example 2.3.5. Let $X := \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2(x+1)) \subset \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$, as depicted in Figure 2.1. Now, at any closed point $p := (a,b)$ of $\mathbb{A}^2_{\mathbb{C}}$, corresponding to the maximal ideal $\mathfrak{m} := (x-a, y-b)$ of $\mathbb{C}[x,y]$, we can see that $\mathfrak{m}/\mathfrak{m}^2 = \mathbb{C}\overline{(x-a)} \oplus \mathbb{C}\overline{(y-b)}$ is a 2-dimensional vector space over the residue field $\mathbb{C}[x,y]/(x-a, y-b) \cong \mathbb{C}$, with basis given by the generators of $\mathfrak{m}$. As such, the tangent space $T_p\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2)$ is a 2-dimensional vector space as well.

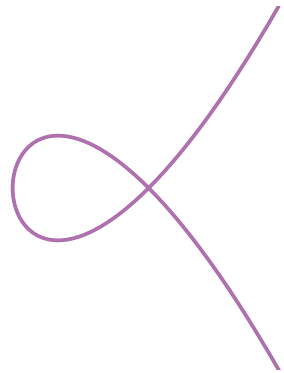


Figure 2.1: The **nodal cubic curve** of Examples 2.3.5, 2.3.7, and 2.3.8.

If $p \in X$, which is to say $b^2 - a^2(a+1) = 0$, we can see by using the usual identification of ideals of a quotient ring with ideals of the original ring that the tangent space T_pX is isomorphic to $\operatorname{Spec}\operatorname{Sym}(\mathfrak{m}/(\mathfrak{m}^2 + I))$ for $I := (y^2 - x^2(x+1))$. Since $y^2 - x^2(x+1) = ((y-b)+b)^2 - ((x-a)+a)^2((x-a)+a+1)$, which is equivalent modulo $\mathfrak{m}^2$ to $2b(y-b) + b^2 - a^2(x-a) - (2a(x-a)+a^2)(a+1) = 2b(y-b) - (3a^2+2a)(x-a)$. If either b or $(3a+2)a$ is nonzero, then, we find that $\mathfrak{m}/(\mathfrak{m}^2+I)$ is a 1-dimensional quotient of $\mathfrak{m}/\mathfrak{m}^2$ and so T_pX is a 1-dimensional subspace of $T_p\mathbb{A}^2_{\mathbb{C}}$, meaning that p is a regular point of X.

Since the equality $3a+2=0$ implies that b is nonzero for (a,b) a point on X, we can see that the only point where this will not hold is the origin in the plane. For this point, we have $I \subset \mathfrak{m}^2$ and so $T_pX = T_p\mathbb{A}^2_{\mathbb{C}}$ is the full 2-dimensional tangent space of the ambient plane, making p a singular point of X. Figure 2.2 illustrates these findings.

Our intuition for the tangent cone, and hence normal cones more broadly, begins by considering the origin in the above example. As we have seen, the Zariski tangent space here is 2-dimensional — that is, every direction in the ambient plane is considered by our definition to be tangent to the curve. This is exactly as it must be, if we want our tangent spaces to be vector spaces, and yet there is something that does not seem to be precisely captured by the tangent space — there are two directions in particular which are somehow

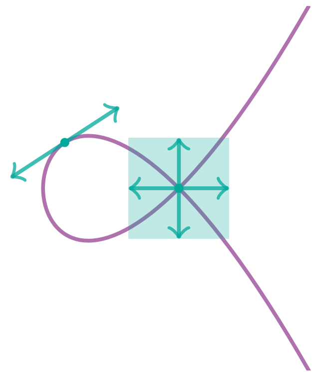


Figure 2.2: The **nodal cubic** with **Zariski tangent spaces** at select points.

"more tangent" to the curve, one corresponding to the limiting tangent line along each branch. This idea is captured by the following definition:

Definition 2.3.6 (e.g., [Eis04; Vak23])**.** Let X be a scheme or complex-analytic space and $x \in X$ a closed point. Then the **tangent cone to X at x** is the normal cone $C_{\{x\}}X$ to the closed subscheme $\{x\}$ in X, as described in Definition 2.3.1. For simplicity, we denote this by $C_x X$.

In the case of a non-closed point, we can make the analogous definition by working in the spectrum of the local ring, but we restrict ourselves to closed points for simplicity. We can see by, for example, considering generators of the maximal ideal $\mathfrak{m}$ in the local ring that the natural map $\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2) \to \operatorname{gr}_{\mathfrak{m}} \mathcal{O}_{X,x}$ is surjective — hence, we have a natural closed inclusion $C_x X \hookrightarrow T_x X$ and so it makes sense to discuss the tangent cone as a subspace of the tangent space. We can see in the case of our prior example that it does in fact capture the limiting tangent lines along each curve branch:

Example 2.3.7 (continuation of Example 2.3.5)**.** As before, we work with the plane curve $X := \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2(x+1)) \subset \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$, this time restricting our attention to the singular point at the origin. Let $\mathfrak{m} := (x,y)$ be the ideal corresponding to this point.

Then, by our prior computations, $\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2) = \mathbb{C}[\bar{x},\bar{y}]$. Letting $I := (y^2 - x^2(x+1))$ as before and setting $R := \mathbb{C}[x,y]/I$, we can see that $\operatorname{gr}_{\mathfrak{m}} R$ is the quotient of this algebra given in each degree d by $\mathfrak{m}^d/(\mathfrak{m}^{d+1} + I \cap \mathfrak{m}^d)$; hence, since, $y^2 - x^2(x+1)$ is in $\mathfrak{m}^2$ and congruent modulo $\mathfrak{m}^3$ to $y^2 - x^2$, we find that $\operatorname{gr}_{\mathfrak{m}} R = \mathbb{C}[\bar{x}, \bar{y}]/(\bar{y}^2 - \bar{x}^2)$. Therefore, this ring's spectrum C_pX is the union in T_pX of the lines $\bar{y} = \bar{x}$ and $\bar{y} = -\bar{x}$, as illustrated in Figure 2.3.

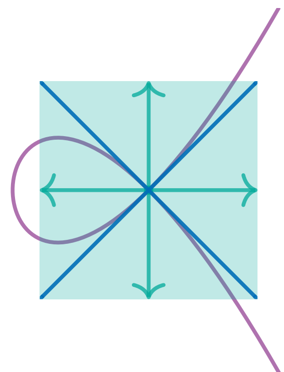


Figure 2.3: The **tangent cone** to the **nodal cubic** in the **tangent space** at the origin.

To see why this should be true, we turn to our deformation to the tangent cone, as given in Definition 2.3.2. We will again discuss our chosen example, the nodal cubic, although the intuition we develop will largely apply in general.

Example 2.3.8 (continuation of Examples 2.3.5 and 2.3.7)**.** Again we let $R := \mathbb{C}[x, y]/(y^2 - x^2(x+1))$, $X := \operatorname{Spec} R \subset \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ and $\mathfrak{m} := (x, y)$. Then the deformation to the tangent cone at the origin is the map of ring spectra induced by $\mathbb{C}[t] \to \mathcal{R}(R, \mathfrak{m}) := R[t, t^{-1}x, t^{-1}y] \subset R[t^{\pm 1}]$.

To see why the tangent cone, which is the fiber of this map over $t = 0$, captures the limiting tangent lines at the origin, we proceed as follows. Regard $\mathcal{R}(R, \mathfrak{m})$ as the quotient of $R[t, \tilde{x}, \tilde{y}]$ by the kernel of the map $\mathbb{C}[t, \tilde{x}, \tilde{y}] \to R[t^{\pm 1}]$ taking $\tilde{x}$ to $t^{-1}x$ and $\tilde{y}$ to $t^{-1}y$. This gives a closed embedding of the deformation's total space into $\mathbb{A}^3_{\mathbb{C}}$; consider the fibers of this embedding over the points of $\operatorname{Spec} \mathbb{C}[t]$.

Over any point other than the origin, as we have discussed, the fiber of the deformation is simply the plane curve $\operatorname{Spec} R := \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2(x+1))$. However, if we use the coordinates $\tilde{x}$ and $\tilde{y}$ for the plane instead of x and y, the embedding into the plane changes as we vary t. Starting at $t = 1$, for example, we find that our new coordinates $\tilde{x}$ and $\tilde{y}$ agree with the old ones, but, e.g., shrinking t moves every point with fixed (x, y) coordinates farther from the origin — for instance, at $t = 1$ the point $(x, y) = (1, \sqrt{2})$ on our curve has $(\tilde{x}, \tilde{y})$ coordinates $(1, \sqrt{2})$, but at $t = \frac{1}{2}$ the same (x, y) pair gives us $(\tilde{x}, \tilde{y}) = (2, 2\sqrt{2})$. That is, since $(\tilde{x}, \tilde{y}) = (t^{-1}x, t^{-1}y)$, changing to our new coordinated system "zooms in by a factor of t^{-1}".

The fiber over the origin, then, can be thought of as the result of "zooming in infinitely far", and naturally this straightens each curve branch into its corresponding tangent line. This process is depicted in Figure 2.4.

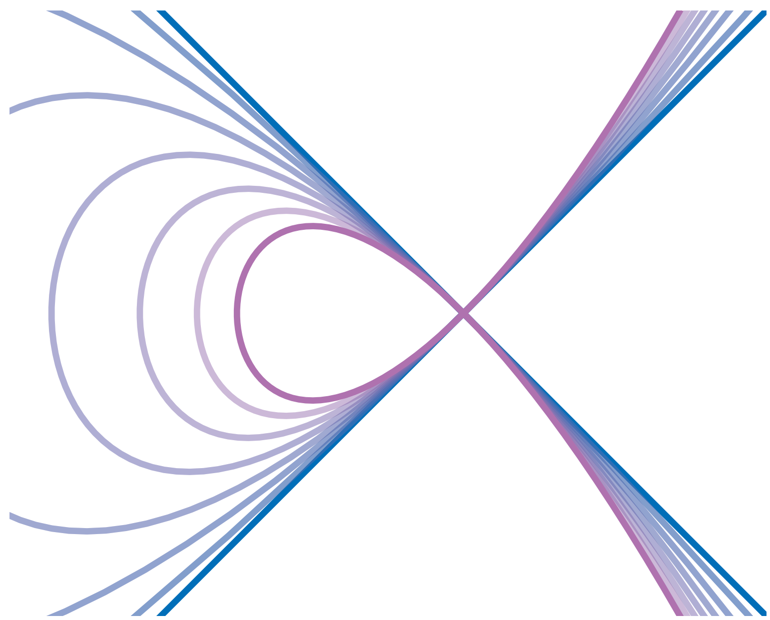


Figure 2.4: The **nodal cubic** deforming to its **tangent cone** at the origin through zooming in.

Hence we see that the tangent cone is the result of "zooming in so far that the space is straightened into a cone", an intuition which will hold in general — note, however, that despite this description the tangent cone does not truly sit inside the hierarchy of "neighborhoods" described in Subsection 2.1.2, since the natural map $C_xX \to X$ factors through x and hence does not meaningfully witness the relationship between the structure of C_xX and the surroundings of x in X. Instead, it is perhaps best to think of the tangent cone as a simplification of the spectrum of the completed local ring which sits somehow "beside" it, in the sense that this ring spectrum deforms to the tangent cone — note that passing to the completion of the local ring does not alter the tangent cone since the inclusions of the infinitesimal neighborhoods factor through the completion spectrum and the tangent cone depends only on the infinitesimal neighborhoods. Indeed, we can see that the degree-d part of the associated graded ring defining the tangent cone is the kernel of the ring map corresponding to the inclusion from the $(d-1)$st-order infinitesimal neighborhood into the dth-order infinitesimal neighborhood — that is, just as $\mathfrak{m}/\mathfrak{m}^2$ captures exactly the first-order derivative information of functions at the point corresponding to $\mathfrak{m}$, $\mathfrak{m}^d/\mathfrak{m}^{d+1}$ captures the dth-order derivative information. Hence the tangent cone and the completed local ring provide two different ways of encapsulating the derivative information of all orders in a single algebro-geometric object.

2.3.2 Intuition for the Normal Cone

Now we adapt these observations for normal cones of arbitrary closed subschemes or complex-analytic subspaces. We begin with an analogue for the Zariski tangent space:

Definition 2.3.9 (e.g., [Har77; Vak23])**.** Let $i : Y \hookrightarrow X$ be a closed inclusion of schemes or complex-analytic spaces, with $\mathcal{I}$ the sheaf of ideals on X cutting out Y. Then $\mathcal{I}/\mathcal{I}^2$ is called the **conormal sheaf of Y in X**; we call the corresponding linear space N_YX over Y, given by $\underline{\operatorname{Spec}}\operatorname{Sym}(i^*\mathcal{I}/\mathcal{I}^2)$ in the algebraic setting and $\operatorname{Specan}\operatorname{Sym}(i^*\mathcal{I}/\mathcal{I}^2)$ in the analytic one, the **normal space to Y in X**.

Note that, as long as $\mathcal{O}_X$ is coherent — in particular, if X is a Noetherian scheme or complex-analytic space — the conormal sheaf will be as well. In the case of a closed embedding of complex manifolds, this definition recovers the usual normal bundle from differential geometry; however, in general $\mathcal{I}/\mathcal{I}^2$ need not be locally free on Y, so N_YX is a linear fiber space with potentially varying fiber dimension rather than a vector bundle. We will discuss the circumstances under which the normal space is a vector bundle in

Subsection 2.4.2.

The first intuition we might draw from the manifold case is that, loosely, the normal space captures information about "tangent directions in X perpendicular to Y", just as the Zariski tangent space captures information about all directions tangent to X at a given point, with the caveat that we do not work with a metric and so the fiber of $N_Y X$ over a given point y of Y is more akin to a quotient than a subspace of $T_y X$. However, even this is somewhat misleading — although there does exist a natural map $T_y X \to N_Y X|_y$, it need not be surjective in general, as the following example shows.

Example 2.3.10. Let $X := \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, $I := (x, y)^2 = (x^2, xy, y^2)$, and $Y := \operatorname{Spec} \mathbb{C}[x, y]/I$. Then, letting p be the origin in X, we can see that the tangent space $T_p X = \operatorname{Spec} \operatorname{Sym}((x, y)/(x, y)^2) = \operatorname{Spec} \operatorname{Sym}(\mathbb{C}\bar{x} \oplus \mathbb{C}\bar{y})$ is a plane, as usual. On the other hand, $N_Y X$ is the spectrum of the symmetric algebra over $\mathbb{C}[x, y]/I$ of $I/I^2 = (x^2, xy, y^2)/(x^4, x^3y, x^2y^2, xy^3, y^4)$. Letting $g_0 := \overline{x^2}$, $g_1 := \overline{xy}$, and $g_2 := \overline{y^2}$, we can see that this algebra is given by $(\mathbb{C}[x, y]/I)[g_0, g_1, g_2]/(yg_0 - xg_1, yg_1 - xg_2, {g_1}^2 - g_0 g_2)$. Hence $N_Y X|_p$ is given by $\operatorname{Spec} \mathbb{C}[g_0, g_1, g_2]/({g_1}^2 - g_0 g_2)$. The natural map $T_p X \to N_Y X|_p$ arises from the inclusion $I \hookrightarrow (x, y)$; composing this with the quotient $(x, y) \to (x, y)^2$ gives a map which sends each element of I^2 and of $(x, y)I$ to zero, so we obtain a map $I/(I^2 + (x, y)I) \to (x, y)/(x, y)^2$ inducing the natural map $T_p X \to N_Y X|_p$ under $\operatorname{Spec} \operatorname{Sym}(-)$. Since $I \subseteq (x, y)^2$, however, we can see that the module map is the zero map, so $T_p X$ is sent to the origin in $N_Y X|_p$.

Hence the normal space in fact captures something slightly subtler than just perpendicular tangent directions — some amount of higher-order information is included as well.

As with the relation between the tangent cone and tangent space described in Subsection 2.3.1, the normal cone, as given in Definition 2.3.1, has a natural closed inclusion $C_Y X \hookrightarrow N_Y X$ over Y into the normal space, given by the natural surjection $\operatorname{Sym}(i^*\mathcal{I}/\mathcal{I}^2) \to i^* \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$ of $\mathcal{O}_Y$-algebras. We should thus likewise think of $C_Y X$ as a refinement of $N_Y X$ giving, roughly, information about the perpendicular directions which "actually come from X". Similarly, we think of the deformation to the normal cone given in Definition 2.3.2 as "straightening X out in directions perpendicular to Y" — this intuition is particularly clear when X and Y are smooth, and so we can locally think of the deformation as coming from a parameterized change of coordinates as in Example 2.3.8. For an illustration in a slightly less mundane case, consider the following modification of Examples 2.3.5, 2.3.7, and 2.3.8:

Example 2.3.11. Let $R := \mathbb{C}[x, y, z]/(y^2 - x^2(x + z))$, so that $X := \operatorname{Spec} R$ is a closed subscheme of $\mathbb{A}^3_{\mathbb{C}}$. Note that the intersection of X with the plane $\{z = 1\}$ gives the nodal cubic of the previous examples, and more generally setting z equal to values other than zero gives us a family of nodal cubics which degenerates to the cusp $\operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3)$ at $z = 0$. Let $Y := \{x = y = 0\}$ be the z-axis in X. Then, if we let $I := (x, y)R$ be the ideal cutting Y out in X, we find that $\operatorname{gr}_I R = (\mathbb{C}[z])[\bar{x}, \bar{y}]/(\bar{y}^2 - z\bar{x}^2)$. Thus we can verify that, in this specific case, the fiber of the normal cone $C_Y X$ over any closed point $\{z = z_0\}$ of the z-axis is simply the tangent cone to the plane curve given by $X \cap \{z = z_0\}$ at the origin. Likewise, the deformation to the normal cone of Definition 2.3.2 can in this instance be interpreted in terms of taking a change of variables to "zoom in with respect to x and y" while leaving z fixed.

Hence, in nice cases, we can think of the normal cone as giving a relative version of the tangent cone, although, as Example 2.3.10 demonstrates, this is not the whole story.

In Subsection 2.3.1, we related the tangent cone at a closed point to the infinitesimal neighborhoods of Definition 2.1.3. As it turns out, this can be done for the normal cone more generally, once we have defined an appropriate notion of infinitesimal neighborhoods of a closed subscheme:

Definition 2.3.12 (e.g., [Har77])**.** Let $i : Y \hookrightarrow X$ be a closed inclusion of schemes or complex-analytic spaces, with $\mathcal{I}$ the sheaf of ideals on X cutting out Y. Then, for any integer $k \geq 0$, the **kth-order infinitesimal neighborhood** of Y in X is the closed subspace of X cut out by the ideal sheaf $\mathcal{I}^{k+1}$.

These subspaces clearly have the same underlying set as Y, but provide successively more infinitesimal information in directions outside of Y as we thicken them by increasing k. As in the case of the tangent cone, the graded pieces of the algebra sheaf defining $C_Y X$ capture exactly the differences between successive infinitesimal neighborhoods, and so the normal cone as a whole encapsulates the data of all infinitesimal neighborhoods — attempting to do this by analogy to the completion, on the other hand, leads to the notion of **formal schemes** (e.g., Section II.9 of [Har77]), which we will not pursue here.

As an example of this principle, we obtain the following straightforward result on the flatness of infinitesimal neighborhoods:

Proposition 2.3.13 ([Hof]). *Let $\pi : X \to S$ be a map of schemes or complex-analytic spaces and $Y \subseteq X$ the closed subspace cut out by a sheaf of ideals $\mathcal{I}$. Then the normal cone $C_Y X$ is flat over S if and only if the k-th order infinitesimal neighborhoods of Y in X are flat over S for all integers $k \geq 0$.*

Proof By Proposition 2.2.19 in the complex-analytic case and the definition of flatness in the algebraic case, the flatness of the normal cone over S is equivalent to that of the associated graded sheaf of algebras $\operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$ over $\pi^{-1}\mathcal{O}_Y$. Since the tensor product distributes over direct sums, the flatness of the normal cone over Y is hence equivalent to that of $\mathcal{I}^k/\mathcal{I}^{k+1}$ over $\pi^{-1}\mathcal{O}_Y$ for all $k \geq 0$, which is to say of the flatness of $D_k := (\mathcal{I}^k/\mathcal{I}^{k+1})_p$ over $R := \mathcal{O}_{Y,\pi(p)}$ for all $k \geq 0$ at each point p of X. Hence we must show that this is equivalent to the flatness of $J_k := (\mathcal{O}_X/\mathcal{I}^{k+1})_p$ over R for each $k \geq 0$ at each such point; therefore, we fix such a p for the remainder of the proof.

In general, the flatness of a module M over R is equivalent to the vanishing of $\operatorname{Tor}_1^R(M, N)$ for all R-modules N and hence of $\operatorname{Tor}_i^R(M, N)$ for all integers $i \geq 1$ and all R-modules N. We can then prove each direction of the equivalence by applying the long exact sequence in Tor to the short exact sequences

$$0 \to D_k \to J_k \to J_{k-1} \to 0$$

for $k \geq 0$; this gives us the exactness of the sequences

$$\mathrm{Tor}_2^R(J_{k-1}, N) \to \mathrm{Tor}_1^R(D_k, N) \to \mathrm{Tor}_1^R(J_k, N) \to \mathrm{Tor}_1^R(J_{k-1}, N)$$

for any R-module N. It is then immediate that the flatness of the infinitesimal neighborhoods over S, which is to say of the J_k over R, implies that of the normal cone.

The reverse implication can be proven by induction on k — suppose that the normal cone is flat over S, so that all of the D_k are flat over R. Then, since $J_0 = D_0$, J_0 is flat over R as well. Now suppose that J_{k-1} is flat over R — since D_k is as well by hypothesis, the exact sequence above gives us $\mathrm{Tor}_1^R(J_k, N) = 0$ for all R-modules N, proving the result.

This inspires us to make the following definition:

Definition 2.3.14. Let $\pi : X \to S$ be a map of schemes or complex-analytic spaces and $Y \subseteq X$ a locally closed subspace. Then we say that **the embedding** $Y \hookrightarrow X$ **is flat over** S if every infinitesimal neighborhood of Y in X is flat over S (or, equivalently, $C_Y X$ is).

We now have the following result on the behavior of the normal cone under pullback:

Lemma 2.3.15 ([Hof])**.** *Let $\pi : X \to S$ be a map of schemes or complex-analytic spaces and $Y \subseteq X$ a locally closed subspace. Let $\phi : S' \to S$ a map of schemes or complex-analytic spaces respectively and suppose either that the embedding $Y \hookrightarrow X$ is flat over S or that ϕ itself is flat. Then the formation of the normal cone commutes with the pullback — that is, $\phi^*(C_Y X) = C_{\phi^* Y}(\phi^* X)$.*

The proof is straightforward and can be found in [Hof].

We conclude by noting that the conormal sheaf and normal space of Definition 2.3.9 give rise to an alternate definition of the relative cotangent sheaf and relative tangent space introduced in Definition 2.2.12:

Proposition 2.3.16 (e.g., [Har77; Loo84]). *Let $X \to Y$ be a map of schemes or complex-analytic spaces. Then, if we let $\Delta : X \to X \times_Y X$ be the diagonal map, Δ is a locally closed embedding. Hence it is cut out by an ideal sheaf $\mathcal{I}$ in some open subset U of $X \times_Y X$; calling the factorization $X \to U$ of Δ through the open subset Δ as well in a slight abuse of notation, we then have $\Omega_{X/Y} \cong \Delta^*(\mathcal{I}/\mathcal{I}^2)$ and so the relative tangent space of X over Y is simply the pullback of the normal space to the diagonal.*

In particular, this makes the coherence of $\Omega_{X/Y}$ clear if the spaces involved are complex-analytic or are locally Noetherian schemes.

2.4 Algebro-Geometric Applications of Normal Cones

We conclude the chapter by surveying some uses of normal cones in algebraic geometry. We would be remiss not to mention the importance of these objects to intersection theory, for which see [Ful98] — however, this is well-known, and we will focus instead on giving reinterpretations of basic constructions and results in terms of normal cones.

2.4.1 Flatness

As we have already discussed in Subsection 2.1.2, flatness is the usual notion of what it means for a map of schemes or complex-analytic spaces to define a "deformation" or "family" of objects, and plays a central role in many algebro-geometric constructions. Despite this, its Definition 2.1.4 is stated very algebraically, and the full geometric consequences can be notoriously difficult to parse.

However, as discussed in the Appendix of [Hir77], there is an equivalent description of flatness which is purely geometric, at least in the case of complex-analytic spaces or locally Noetherian schemes. This is given by the following theorem:

Theorem 2.4.1 (e.g., [Hir77] or Theorem 22.3 of [Mat89]). *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of Noetherian local rings, $\kappa = R/\mathfrak{m}$ the residue field, $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$ the corresponding schemes with closed points x and y respectively, and*

$f : X \to Y$ the corresponding map of schemes. Then S is flat over R if and only if the natural map $\operatorname{gr}_{\mathfrak{m}} R \otimes_{\kappa} S/\mathfrak{m}S \to \operatorname{gr}_{\mathfrak{m}S} S$ is an isomorphism — that is, if and only if $C_{X_y}X \cong X_y \times_y C_yY$ by the natural map for $X_y := f^{-1}(y)$ the fiber over the closed point.

Since flatness is defined in terms of maps of local rings, this is to say that, in either of the contexts we are interested in, flatness of a map can be defined as the **triviality of the normal cone to each fiber over the tangent cone to the corresponding point in the base** — the above theorem establishes this fact locally at each point of the source, and it is not difficult to verify that the trivialization glues together along each fiber. This is to say that, although flat maps need not be locally trivial in the manner of fiber bundles, they are precisely the maps which become locally trivial when "things are straightened out in the horizontal directions" in the sense given by taking normal cones.

Armed with this description, we can now prove Proposition 2.1.6:

Proof of Proposition 2.1.6 It is enough to observe that the map of associated graded rings with respect to the maximal ideal $(x_1, \ldots, x_a, y_1, \ldots, y_b)$ of the domain induced by $\mathbb{C}\{x_1, \ldots, x_a\}[y_1, \ldots, y_b]_{(x_1,\ldots,x_a,y_1,\ldots,y_b)} \to \mathbb{C}\{x_1, \ldots, x_a, y_1, \ldots, y_b\}$ is an isomorphism, since both associated graded rings are simply $\mathbb{C}[\bar{x}_1, \ldots, \bar{x}_a, \bar{y}_1, \ldots, \bar{y}_b]$ and the induced map is the identity. Hence, by Theorem 2.4.1, the map is flat, and so for faithful flatness it is enough to verify surjectivity on closed points, which is immediate for a local map of local rings such as this one.

That is, the faithful flatness follows since both rings' spectra have the same tangent cone at the closed point.

2.4.2 Regular Embeddings

As we have mentioned in Subsection 2.3.2, the intuition for the normal cone is comparatively simple for an embedding $Y \hookrightarrow X$ of complex-analytic manifolds or smooth varieties over $\mathbb{C}$ — in this case, $C_YX = N_YX$ is the vector bundle over Y given as the quotient of the restricted tangent bundle $TX|_Y$ by the tangent bundle TY. We can then ask which closed embeddings of schemes and complex-analytic spaces "look like embeddings of manifolds" in the sense that the normal cone is a vector bundle.

Definition 2.4.2 (e.g., [Ful98; Vak23])**.** Let $i : Y \hookrightarrow X$ be a closed embedding of locally Noetherian schemes or complex-analytic spaces, with $\mathcal{I}$ the corresponding ideal sheaf. We say that i is a **regular embedding** if, for each point $y \in Y$, $\mathcal{I}_y$ is generated by a finite-length **regular sequence** (for which see, e.g., Chapter 17 of [Eis04]) in $\mathcal{O}_{X,y}$.

Note that it is equivalent to ask that $\mathcal{I}$ be generated by a regular sequence in an open neighborhood of each point (e.g., Exercise 9.5.G of [Vak23]). Such embeddings are of great practical importance in algebraic geometry — aside from their use in intersection theory, for which see [Ful98], they also give rise to much-used notions such as **depth** and **Cohen-Macaulayness**, for which see, e.g., [Eis04]. The following result characterizes them in geometric terms by showing that they are exactly those for which the normal cone is a vector bundle:

Proposition 2.4.3 ([Ree57]; also, e.g., Exercise 17.16 of [Eis04])**.** *A closed inclusion $Y \hookrightarrow X$ of locally Noetherian schemes or complex-analytic spaces is a regular embedding if and only if $C_Y X$ is a vector bundle over Y. Moreover, a given choice of local generators for the ideal sheaf of Y in X will form a regular sequence exactly when the generators' initial forms give a local basis of sections for $C_Y X$.*

In particular, our notion of the regularity of a scheme at a point from Definition 2.3.4 can be rephrased. Let $(R, \mathfrak{m}, \kappa)$ be a local ring, setting $X := \operatorname{Spec} R$ and $x := \operatorname{Spec} \kappa$, and consider the surjection $\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2) \to \operatorname{gr}_{\mathfrak{m}} R$ of κ-algebras. Since its kernel is generated in degrees 2 and higher by construction, we can see that $C_x X$ is a vector space if and only if $C_x X = T_x X$. On the other hand, since $T_x X$, as an affine space over κ, is reduced and irreducible, we can see that its dimension will be the same as that of the closed subscheme $C_x X$ if and only if, again, the two are again equal. By the flatness of the deformation to the normal cone (Proposition 2.3.3), however, $\dim C_x X$ is precisely the local dimension of X by, e.g., Theorem 10.10 of [Eis04].

Hence a scheme is regular at a point precisely when the inclusion of the point is a regular embedding — for non-closed points, we make sense of this by working with the embedding into the spectrum of the corresponding local ring. Together with Theorem 2.4.1, this gives us a powerful dimensional characterization of flatness for maps of regular schemes:

Theorem 2.4.4 (e.g., Theorem 18.16 of [Eis04])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of regular local rings. Then S is flat over R if and only if* $\dim S/\mathfrak{m}S = \dim S - \dim R$*.*

The proof and most general statement of this theorem use the machinery of Cohen-Macaulay rings, so we omit them. However, we do note that it generalizes to the following important class of local rings.

Definition 2.4.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say that R is a **local complete intersection ring** if the spectrum $\operatorname{Spec} \hat{R}$ of its completion can be realized as the special fiber of a flat map of regular local rings.

Proposition 2.4.6. *Theorem 2.4.4 remains true if we relax our requirement on S so that it is required only to be a local complete intersection ring, not necessarily a regular one.*

2.4.3 Blowups

The following construction is ubiquitous in algebraic geometry:

Definition 2.4.7 (e.g., [Har77; Eis04; Vak23])**.** Let $i : Y \hookrightarrow X$ be a closed inclusion of schemes or complex-analytic spaces, with $\mathcal{I}$ the ideal sheaf cutting it out. Then we define the **blowup algebra sheaf** of Y in X by

$$B_{\mathcal{I}}\,\mathcal{O}_X := \bigoplus_{k=0}^{\infty} \mathcal{I}^k$$

with the obvious multiplication making this a sheaf of graded algebras. In the algebraic and complex-analytic cases respectively we define by $\operatorname{Bl}_Y X := \underline{\operatorname{Proj}}\, B_{\mathcal{I}}\,\mathcal{O}_X$ and $\operatorname{Bl}_Y X := \operatorname{Projan} B_{\mathcal{I}}\,\mathcal{O}_X$ the **blowup** of X along Y.

To understand the geometric intuition for this construction, we note that, over $X \setminus Y$, $B_{\mathcal{I}}\,\mathcal{O}_X \cong \mathcal{O}_X[t]$, since $\mathcal{I}|_{X\setminus Y} \cong \mathcal{O}_{X\setminus Y}$, and hence $\mathrm{Bl}_Y X \to X$ restricts over this subspace to an isomorphism. On the other hand, the restriction $i^* B_{\mathcal{I}}\,\mathcal{O}_X$ over Y is exactly the associated graded $i^* \operatorname{gr}_{\mathcal{I}} \mathcal{O}_X$ — that is, the restriction $(\mathrm{Bl}_Y X)|_Y$ of the blowup over Y itself, which is called the **exceptional divisor**, is exactly the projectivized normal cone to Y in X.

Therefore, if we keep to our rough intuition of the normal cone as capturing "directions perpendicular to Y in X" — and hence the projectivized normal cone as having points corresponding to such "directions" — the operation of taking the blowup along Y can be understand as replacing Y in X with its projectivized normal cone in such a way that the complement $X \setminus Y$ is glued onto this space "according to the limiting directions along which it originally approached Y in X".

Of the many contexts in which blowups arise, particularly relevant to us is the notion of **resolution of singularities**. In brief, this is the process of attempting to eliminate the singular points of a space by repeatedly blowing up along closed subspaces — we will return to the concept in Subsection 3.1.1.

2.4.4 Normalization and Singularities in Codimension 1

It is a well-known algebro-geometric fact that **normalization** (for which see, e.g., Section 10.7 of [Vak23]) "resolves singularities in codimension 1" — that is, that the locus of singular points of, say, a finite-type integral normal scheme over a perfect field is of codimension at least 2. We will explain, briefly, how to prove this by means of the normal cone.

To do so, we first need the following lemma, which relies on the standard notions of the **finiteness** and **birationality** of morphisms — these can be found in, e.g., [Har77; Vak23].

Lemma 2.4.8. *Let k be a perfect field, X an integral finite-type k-scheme, and $\nu : \tilde{X} \to X$ the normalization. For Y another integral finite-type k-scheme, suppose that we have a* *finite birational morphism $\phi : Y \to X$. Then ν factors uniquely through ϕ.*

Proof To begin with, we note that the normalization is an affine morphism by definition and ϕ is as well by virtue of being finite, we can prove this on the level of rings by working locally on X. Thus, it suffices to prove that an inclusion $R \hookrightarrow S$ of finitely-generated integral domains over k which makes S a finite R-module and induces an isomorphism on fields of fractions gives rise to a unique map $S \hookrightarrow \bar{R}$ whose composition with the original yields the natural inclusion $R \hookrightarrow \bar{R}$, where $\bar{R}$ denotes the integral closure of R in its field of fractions K.

This follows immediately by noting that we have a natural identification of S with a subring of the field of fractions K of R and thus that R, S, and $\bar{R}$ are all subrings of K — since S is a finite R-module, each of its elements must be integral over R and hence we have $R \subseteq S \subseteq \bar{R}$ in K. The uniqueness follows by noting that maps from subrings of K containing R are determined by their restriction to R.

Thus, at least in this context, X being a normal scheme is equivalent to there being no nontrivial finite birational map to X. By the finiteness of integral closure in this context (e.g., Theorem 10.7.3 of [Vak23]), the property in the preceding lemma also characterizes the integral closure up to unique isomorphism for such schemes.

We are now prepared to verify that normalization resolves singularities in codimension one:

Theorem 2.4.9. *Let k be a perfect field and X an integral finite-type k-scheme which is, moreover, normal. Then the singular locus of X has codimension at least 2.*

Proof We will see by Proposition 3.1.3 that the singular locus is closed in X — let Y denote the subscheme of X given by taking this set with its reduced scheme structure. By Theorem 2.2.14, the complement of Y is dense in X and so its dimension is everywhere at most $n-1$ for $n := \dim X$.

Suppose toward a contradiction that Y has an irreducible component whose dimension achieves this upper bound. Then, again by Theorem 2.2.14, an open dense subset of this irreducible component is smooth — hence, since normalization commutes with localization, we can suppose by restricting to an open subset of X that Y is smooth of codimension 1. We now claim that, at least if we restrict to a further dense open subset of X, the blowup map $\mathrm{Bl}_Y X \to X$ is a nontrivial finite birational map to X — this will prove the claim by Lemma 2.4.8.

Birationality is immediate from our prior discussion of the blowup in Subsection 2.4.3 since Y has dense complement in X. To show finiteness, we note that the blowup map is projective by construction and hence proper — thus, by Theorem 28.6.2 of [Vak23] and the fact that the blowup is a finite-type map in this case, it is enough to verify that its fibers are finite as sets. This is immediate away from Y. By the flatness of the deformation to the normal cone (Proposition 2.3.3) and the fact that X is of pure dimension n, we see that every component of $C_Y X$ must be n-dimensional — therefore, by restricting over a further open subset as necessary, we find that the fibers of $C_Y X \to Y$ have pure dimension 1 and so, since the exceptional divisor is the projectivized normal cone, the fibers of $\mathrm{Bl}_Y X \to X$ have pure dimension zero, as desired.

It remains to show that the blowup map is nontrivial. Let $y \in Y$ be a closed point and, working in an affine patch around y, suppose that X is locally given by $\operatorname{Spec} R$, with I the ideal of R cutting out Y and $\mathfrak{m}$ the ideal of R cutting out y. Then we have natural maps $T_y Y \hookrightarrow T_y X \to N_Y X|_y$ corresponding to the exact sequence $I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/(\mathfrak{m}^2 + I) \to 0$ of vector spaces over the residue field $\kappa \cong k$ in the other direction. Since y is a singular point of X, we find that $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) \geq n+1$; on the other hand, since Y is smooth, $\dim_\kappa(\mathfrak{m}/(\mathfrak{m}^2 + I)) = n - 1$. Thus it follows that $\dim_\kappa(I/\mathfrak{m}I) \geq (n+1) - (n-1) = 2$ for each such y.

Now suppose toward a contradiction that the blowup map is trivial. Then the exceptional divisor must simply be Y itself and so $C_Y X$ is necessarily a line bundle over Y embedded in $N_Y X$ — here we can preclude the possibility of embedded components at the zero section of $C_Y X \to Y$ by our prior observation that the normal cone has pure dimension n. In particular, $C_Y X$ is a linear subspace of $N_Y X$ over Y — however, it is clear from the definitions that all of the generators of the ideal cutting it out in this normal space have degree at least 2, so its linearity implies that it is actually equal to $N_Y X$, which is therefore itself a line bundle. Since this contradicts our prior result that the fibers of $N_Y X$ are of dimension at least 2, the result follows.

Chapter 3

Singularities and Stratified Maps

In Chapter 2, we discussed ways of working with complex-analytic spaces by, essentially, treating them like schemes — as we saw, most algebro-geometric constructions and results admit some adaptation to the analytic setting, in no small part because of the local equivalence of Proposition 2.1.2. However, this is not the only perspective we can take — many algebro-geometric concepts, after all, arise as analogues of ideas from differential geometry, and, since the underlying sets of complex-analytic spaces are locally given by vanishings of collections of analytic functions in $\mathbb{C}^n$, we can also attempt to apply techniques from differential geometry more directly.

Here we will develop some of the machinery necessary to do so. Section 3.1 will discuss in more detail the **singularities** of schemes and complex-analytic spaces before introducing **stratifications**, tools which let us study even singular spaces in terms of differential geometry. Section 3.2 will elaborate on the relative theory of stratifications by providing additional conditions and results; Section 3.3 will conclude by laying out the proof of a stratification theorem for families of holomorphic maps.

3.1 Stratifications and Whitney Conditions

As mentioned, we begin with a discussion of the singularities of spaces and maps, followed by some basic definitions and results from stratification theory.

3.1.1 Motivation: Smoothness and Singularities

Using differential geometry to study complex-analytic spaces will, of course, be vastly simpler when the spaces are themselves manifolds with no additional non-reduced structure. We have already seen two algebro-geometric definitions which purport to capture this notion — the **smoothness** of Definition 2.2.13 and the **regularity** of Definition 2.3.4, based on the notions of a **tangent space** given by Definitions 2.2.12 and 2.3.4 respectively. To relate the two notions, we begin by noting that these constructions agree in the cases we will be interested in:

Proposition 3.1.1. *Let X be a scheme over a perfect field k or a complex-analytic space, and $x \in X$ a point which, in the algebraic case, we suppose to have residue field k. Then there is a natural isomorphism $TX|_x \cong T_xX$ from the restriction of TX over x to the Zariski tangent space.*

The proof in the algebraic case can be found at [Sta23, Tag 0B28]; the complex-analytic case can be verified explicitly using the local embeddings of X into Euclidean space and standard properties of the cotangent sheaf (e.g., [Tei77]).

This leads us to the following theorem:

Theorem 3.1.2 (e.g., [Vak23])**.** *Let k be a perfect field. Then every smooth k-scheme is regular and every finite-type regular k-scheme is smooth. For complex-analytic spaces, smoothness and regularity are equivalent, and both conditions are moreover equivalent to being locally isomorphic to $\mathbb{C}^n$ for some n.*

In the complex-analytic case, the equivalence of smoothness and regularity follows from Proposition 3.1.1 by using **Nakayama's Lemma** (e.g., Corollary 4.8 of [Eis04]) to extend bases of T_xX, although some care must be taken in dealing with the possibility of lower-dimensional components; for final statement, see, e.g., Section 2.15 of [Fis76].

Hence we can focus on understanding smoothness and regularity by means of the tangent bundle. As a first step in this direction, we have the following **Jacobian Criterion** for checking the smoothness of maps:

Proposition 3.1.3. *Let $\phi : X \to Y$ be a flat map set-theoretically of pure relative dimension d between finite-type schemes over a perfect field or complex-analytic spaces. Then the locus of points in X where ϕ fails to be smooth is exactly the vanishing of the dth* **Fitting ideal sheaf** *(see, e.g., Section 20.2 of [Eis04])* $\mathrm{Fitt}_d(\Omega_{X/Y})$ *of the sheaf of Kähler differentials.*

Proof In these circumstances, smoothness is equivalent to $\Omega_{X/Y}$ being locally free of rank d. Hence, by Proposition 20.6 of [Eis04], the failure of smoothness at points of the subspace cut out by $\mathrm{Fitt}_d(\Omega_{X/Y})$ is immediate. On the other hand, at points outside this subspace, $\Omega_{X/Y}$ is locally generated by d elements, which is to say that it locally admits a surjection from a free sheaf of rank d. We consider such a point $x \in X$, fix such a surjection, and argue that it must be an isomorphism.

To begin with, we note that, in the algebraic case, we can suppose without loss of generality that x is a closed point by passing our hypotheses to a closed point of a small enough open neighborhood of x. Now let $y = \phi(x)$ and note that, since the formation of relative differential commutes with pullback along maps to the codomain (e.g., [Eis04; Tei77]), the restriction of $\Omega_{X/Y}$ to the fiber X_y of X over y gives us simply Ω_{X_y}. By Proposition 3.1.1 and the fact that the dimension of the Zariski tangent space of the fiber at any point is at least d by the dimensionality hypothesis on ϕ, it is then immediate from Theorem 3.1.2 that X_y is smooth at x.

In the algebraic case, the result now follows by [Sta23, Tag 01V8], while in the complex-analytic case it is immediate from the **simplicity theorem** of [Tei77].

Note that, if Y is the spectrum of the ground field in the algebraic case or a reduced point in the complex-analytic case, we recover the usual Jacobian criterion (e.g., Theorem 16.19 of [Eis04]) using the standard **conormal sequence** (e.g., [Eis04; Tei77]).

This result motivates the following definition:

Definition 3.1.4 (cf. [Tei77])**.** Let $\phi : X \to Y$ be a flat map set-theoretically of pure relative dimension d between finite-type schemes over a perfect field or complex-analytic spaces. Then we denote the Fitting ideal sheaf $\mathrm{Fitt}_d(\Omega_{X/Y})$ by $\mathcal{J}_{X/Y}$ (or $\mathcal{J}_\phi$) and call it the **Jacobian ideal sheaf** of X over Y (or of ϕ). The closed subscheme cut out by this sheaf of ideals will be denoted by $\Sigma_{X/Y}$ (or Σ_ϕ) and called the **singular** or **critical locus** of X over Y (or of ϕ).

If Y is the spectrum of the ground field in the algebraic case or a reduced point in the complex-analytic case, we drop Y from the notations and terminology and prefer the word "singular" to "critical".

Away from such singular points, the study of complex-analytic spaces and maps is straightforwardly tractable through classical techniques from differential geometry, as desired, and so it is the singularities which will occupy our attention going forward. One popular approach to dealing with the existence of singular points for a scheme or complex-analytic X is to seek a map from a non-singular space onto X which satisfies certain reasonable hypotheses — this gives rise to the theory of **resolution of singularities**, which has a long and storied history we will not explore in detail. For our purposes, it will be enough to note that, for complex-analytic spaces, resolutions exist by the results of Hironaka:

Theorem 3.1.5 (e.g., [Wło09])**.** *Let X be a reduced complex-analytic space. Then there exist a smooth complex-analytic space $\tilde{X}$ and a proper bimeromorphic map $\tilde{X} \to X$ which is an isomorphism over the smooth part of X.*

In fact, stronger statements can be made about the structure of the desingularization, but we omit them. For our purposes, it will be more fruitful to study smooth complex-analytic spaces which can be embedded in X.

3.1.2 Definitions and First Results

We may seek to understand singular spaces by breaking them into smooth pieces, an approach which gives rise to the theory of stratifications. We will now provide an introduction to this area of study, following the expositions of [Hir77; GM88; Dim92; Tro20].

Definition 3.1.6. Let M be a smooth manifold and $X \subseteq M$ a locally closed subset. A **stratification** of X is a decomposition $X = \bigcup_i X_i$ of X into disjoint locally closed smooth submanifolds X_i, called **strata**, such that:

- Each point of X has a neighborhood which meets only finitely many strata.
- The closure in X of each stratum is a union of strata.

This latter requirement is called the **frontier condition**. Some authors also require that each stratum be connected, while others do not — we will take the former view.

If M is a real-analytic manifold, we say that a stratification is **real-analytic** if the closure of each stratum in A is so. The definition of a **complex-analytic stratification** of a subset in a complex-analytic manifold M is analogous.

If X is a complex-analytic space, we say that a decomposition of the underlying set is a **stratification** if it is so with respect to the local embeddings into Euclidean space.

As discussed, we can now use the techniques of differential geometry to study singular spaces by applying them stratum-wise. However, for many important applications the definition of a stratification does not impose enough control on the relationships between the strata — in particular, it is natural to want the space under consideration to "look the same" from point to point within a stratum, but this is not guaranteed, as the following example shows:

Example 3.1.7 (e.g., [Dim92])**.** Let W be the complex-analytic surface, called the **Whitney umbrella**, in $\mathbb{C}^3$ cut out by the equation $x^2 - y^2 z = 0$. Then, if we let L denote the z-axis in $\mathbb{C}^3$, we can see that $\{W \setminus L, L\}$ is a stratification of W. Then, around points of L other than the origin, we can see by taking a local branch of $\sqrt{z}$ that the triple of complex-analytic spaces $(\mathbb{C}^3, W, L)$ is locally biholomorphic to $(\mathbb{C}^3, \{xy = 0\} \times \mathbb{C}, 0 \times \mathbb{C})$ through a change of coordinates. At the origin, however, this is not true even on the level of topological spaces, since, for example, the local homology of W at the origin does not agree with that of a pair of complex planes meeting in a line.

To rectify this issue, we introduce the following conditions, originally due to Whitney:

Definition 3.1.8 (e.g., [Hir77; GM88; Dim92; Tro20])**.** Let X be a locally closed subset of a smooth manifold M and $\{S_\alpha\}_\alpha$ a stratification of X. Then, if $S_\alpha \subset \overline{S_\beta}$ are strata and s a point of S_α, we say:

- (S_β, S_α) satisfies **Whitney's condition (a)**, or is **Whitney (a)-regular**, at s if, for each sequence x_i of points in S_β approaching s such that the limiting tangent space $T := \lim_{i\to\infty} T_{x_i} S_\beta$ exists, $T_s S_\alpha \subset T$.
- (S_β, S_α) satisfies **Whitney's condition (b)**, or is **Whitney (b)-regular**, at s if, for each choice of sequences s_i in S_α and x_i in S_β approaching s such that the limiting tangent space $T := \lim_{i\to\infty} T_{x_i} S_\beta$ and limiting secant line $\ell := \lim_{i\to\infty} \overline{s_i x_i}$ both exist, $\ell \subseteq T$. Since the statement of this condition no longer uses the smoothness of S_α, we will on occasion speak of Whitney (b)-regularity in situations where the lower "stratum" is potentially singular.

We say that such a pair of adjacent strata satisfies either of these conditions if it does so at each point of the lower-dimensional stratum, and that a stratification does so if each pair of adjacent strata does. A **Whitney stratification** is defined to be a stratification satisfying both conditions.

If X is a complex-analytic space, we again define these conditions on a stratification locally using the local embeddings into Euclidean space.

Note that, in the complex-analytic case, we can check these conditions using holomorphic tangent spaces and complex lines. Moreover, implicit in our statement of the condition (b) is that it does not depend on notion of a "line" — that is, on our choice of local coordinates for M. This can be verified directly by comparing the local foliations arising from different local coordinate systems.

We have stated both conditions since (a)-regularity is easier to understand and will be useful for our discussion of the Thom condition in Section 3.2, but it should be noted that the conditions turn out to be redundant:

Proposition 3.1.9 (e.g., [Tro20])**.** *For any individual pair of strata, Whitney's condition (b) implies Whitney's condition (a).*

As we will see in Subsection 3.1.3, Whitney's conditions are enough to prevent failures of local consistency of the sort arising in Example 3.1.7. Hence, if we are given a singular space and wish to study it by way of differential geometry, using a Whitney stratification to do so is a natural choice — it thus behooves us to establish that these stratifications exist in general. The proof of this fact is in large part a consequence of the following **Generic Whitney Lemma**, which we state in the complex-analytic case; the somewhat subtler real-analytic version is unnecessary for our purposes and can be found in [Hir77].

Lemma 3.1.10 (e.g., [Hir77])**.** *Let M be a complex-analytic manifold, $B \subseteq M$ a connected complex-analytic submanifold, and $A \subseteq \bar{B} - B$ a closed complex-analytic subspace of M. Then there exists a closed complex-analytic subspace $S \subset A$, nowhere dense in A, such that, for each $p \in A$, (B, A) is Whitney (b)-regular in a neighborhood of p if and only if $p \notin S$.*

This is enough to guarantee the existence of complex-analytic Whitney stratifications of complex-analytic subspaces of complex-analytic manifolds:

Theorem 3.1.11 (e.g., [Tei82; Tro20])**.** *Let X be a complex-analytic space. Then X admits a complex-analytic Whitney stratification.*

We can also generalize the idea of a stratification to the setting of maps between complex-analytic spaces.

Definition 3.1.12 (e.g., [Hir77; GM88])**.** Let $\phi : X \to Y$ be a map of locally closed subspaces of smooth manifolds arising from a restriction of a smooth function or a map of complex-analytic spaces. Then a **stratification** of ϕ is given by stratifications of X and Y such that each stratum of X is mapped surjectively and submersively onto some stratum of Y. We say that such a stratification is **$\mathbb{C}$-analytic** or a **Whitney stratification** if each of its constituent stratifications satisfies the respective condition.

Note that, in particular, this makes the image of ϕ a union of strata in Y. Stratifications of maps are useful for much the same reasons stratifications of spaces are; they decompose a map into pieces such that each piece is, from the perspective of differential geometry, "as nice as possible". It behooves us, then, to establish that Whitney stratifications of maps actually exist in good cases:

Theorem 3.1.13 (e.g., [Hir77; GM88]). *Let $\phi : X \to Y$ be a proper map of complex-analytic spaces. Let $P(S, s)$ and $Q(S, s)$ be propositions defined for points s in complex-analytic submanifolds S of X and Y respectively, such that the failure of P or Q on such a submanifold occurs exactly on a locally closed complex-analytic subset of S which is nowhere dense.*

Then ϕ admits a Whitney stratification such that each stratum of X satisfies P at every point and each stratum of Y satisfies Q at every point.

This is slightly more general than the version which is usually stated. Instead of the conditions P and Q, it is typical to specify locally finite collections of complex-analytic subsets which are required to be unions of strata — this corresponds to the propositions "S is either contained in or disjoint from each of the chosen subsets in a neighborhood of s" on X and Y. However, the proof in [Hir77] establishes the more general result with little modification, and the added versatility will be useful to us. For a reasonably comprehensive list of settings other than the complex-analytic one where the analogous theorem holds, see Section I.1.7 of [GM88].

3.1.3 Thom's First Isotopy Lemma and Local Triviality

As one indication of the usefulness of stratified maps, we have the following stratified analogue of the Ehresmann Fibration Theorem, called **Thom's First Isotopy Lemma**:

Lemma 3.1.14 ([Mat70; Mat12]). *Let X be a locally closed subset of a smooth manifold M and $f : M \to N$ a map to another smooth manifold N whose restriction to X is proper. Then, if X has a Whitney stratification such that the restriction of f to each stratum is a smooth submersion, f is a stratified-homeomorphically locally trivial fibration whose trivializations are moreover diffeomorphic along each stratum.*

It is common to omit the statement about stratum-wise diffeomorphism entirely, and state this lemma in terms of topological triviality alone. However, as Goresky and MacPherson note in their statement of the lemma in Section I.1.5 of [GM88], the local trivialization homeomorphisms are indeed smooth along each stratum — in fact, the proof in Mather's notes shows that the inverses are smooth as well, so they are even stratum-wise diffeomorphisms. This is because the trivializations and their inverses arise from the integration of controlled vector fields, which are by definition smooth along each stratum and hence yield flows which are likewise stratum-wise smooth. Thus the proof of Thom's First Isotopy Lemma 3.1.14 gives a stronger result than is usually explicitly stated, even in [Mat70; Mat12].

Thom's First Isotopy Lemma 3.1.14 allows us to show that Whitney regularity precludes bad behavior of the sort we encountered in Example 3.1.7:

Theorem 3.1.15 (e.g., [Tro20]). *Let W be a Whitney-stratified space, $s \in W$ a point, and S the stratum containing it. Then W is locally stratified-homeomorphic around s to the product of S with a stratified space N in such a way that the homeomorphism is diffeomorphic along each stratum. Moreover, N can be taken to be the open cone over a stratified subset of a sphere.*

That is, Whitney stratifications are locally trivial along each stratum, fulfilling our desire for a stratification which "looks the same from point to point within a stratum". Care should be taken as to the nature of the trivializations, however — although they are diffeomorphic along each stratum, Example 4.2.16 of [Tro20] shows that we cannot hope for them to arise from diffeomorphisms of the ambient space in general.

3.2 The Thom Condition and Relative Conormal Space

By Theorem 3.1.13, we know that Whitney stratifications exist at least for proper maps of complex-analytic spaces. However, the Whitney conditions themselves are defined with respect to the source and target spaces independently and, a priori, have little to do with the map between them — indeed, even the requirement of being a stratumwise surjective submersion does not tell us much about the relationship between the map's behaviors over differing strata of the codomain. Here we will introduce a more stringent relative condition which gives finer control over the behavior of the map under consideration.

3.2.1 The Thom Condition

The following relative version of Whitney's condition (a) imposes constraints of the sort we want on the relationships between the various stratumwise fibers:

Definition 3.2.1. [Mat70; Mat12] Let X be a locally closed subset of a smooth manifold M, $\{S_\alpha\}_\alpha$ a stratification of X, and f the restriction to X of a differentiable map from M to another smooth manifold N. Then, if $S_\alpha \subset \overline{S_\beta}$ are strata and s a point of S_α such that $f|_{S_\alpha}$ and $f|_{S_\beta}$ have constant rank near s, we say that (S_β, S_α) satisfies the **Thom (a_f) condition**, or is **Thom (a_f)-regular**, at s if the following holds: For each sequence of point x_i in S_β approaching S such that the limiting vertical tangent space $T_f := \lim_{i\to\infty} T_{s_i}(S_\beta \cap f^{-1}(f(s_i)))$ exists, $T_s(S_\alpha \cap f^{-1}(f(s))) \subseteq T_f$. We say that the pair (S_β, S_α) satisfies this condition if it does so at every point of S_α and f has constant rank everywhere on S_β.

In the case of a map of complex-analytic spaces, we define these conditions locally as usual. By a **Thom stratification** of a map f we will mean a Whitney stratification which moreover has the property that every pair of adjacent strata in the domain is Thom (a_f)-regular.

This condition is strong enough to be quite useful in controlling the behavior of fibers of stratified maps, as we will see in Chapter 4; however, it is also strong enough to no longer be achievable in general, as the following example shows.

Example 3.2.2 ([Hir77])**.** The natural projection of the blowup of a point in $\mathbb{C}^n$ does not admit a Thom stratification for $n \geq 2$, since the fibers away from the blown-up point all have zero-dimensional tangent spaces and the fiber over the blown-up point is not finite.

It is then natural to ask under what circumstances such a stratification can be found. In [Hir77], Hironaka proves a rather technical result in this direction using relative Nash-style modifications, which implies the existence of Thom stratifications for proper complex-analytic maps to complex-analytic curves. This result will be of interest to us, but we defer it to Subsection 3.2.3, where we will prove it using more modern machinery.

The Thom condition gives rise to a relative version of Thom's First Isotopy Lemma 3.1.14, called **Thom's Second Isotopy Lemma** — the details can be found in [Mat70; Mat12].

3.2.2 The Relative Conormal Space

The Thom condition of Definition 3.2.1 is defined in terms of limits of sequences of tangent spaces along smooth fibers of a function f, and we can just as well understand this dually in terms of the limiting cotangent vectors vanishing on these tangent spaces. Hence we can study the Thom condition in terms of the following object, as in, e.g., [LT88; BMM94; GR19]:

Definition 3.2.3 (e.g., [GM18])**.** Let $\Phi : X \to Y$ be a map between smooth complex-analytic spaces which is a submersion on an open dense subset U of X. Then the **relative conormal space** of Φ is the closed subset of the cotangent bundle T^*X given by

$$T^*_\Phi X := \overline{\{(x, \eta) \in T^*U \mid \eta(\ker D\Phi|_x) = 0\}},$$

considered as a complex-analytic space via the reduced structure.

This is particularly easy to understand in the case where Y is 1-dimensional, say $Y = \mathbb{C}$; in this case $T^*_\Phi X|_U$ will be the bundle of cotangent lines spanned over each point by the gradient $d\Phi$, and so each fiber of $T^*_\Phi X$ will simply be the union of limiting gradient lines of Φ at the point in question.

Observe that, since T^*X is a vector bundle, and hence a complex-analytic cone, over X and $T^*_\Phi X|_U$ is invariant under the scaling action in $T^*X|_U$, its closure will be invariant under scaling as well and so $T^*_\Phi X$ is a complex-analytic subcone of T^*X. It is then natural to seek to understand it in terms of the corresponding sheaf of algebras, which is given by the following proposition:

Proposition 3.2.4 ([Hof]). *Let $\Phi : X \to Y$ be a map between smooth complex-analytic spaces which is a submersion on an open dense subset U of X. Then the inclusion $T^*_\Phi X \hookrightarrow T^*X$ corresponds, under the anti-equivalence between complex-analytic cones over X and graded sheaves of $\mathcal{O}_X$-algebras given in Theorem 2.2.16, to the surjection from* $\mathrm{Sym}(\Theta_X)$ *to the sheaf of $\mathcal{O}_X$-algebras which is the image of the map* $\mathrm{Sym}(D\Phi) : \mathrm{Sym}(\Theta_X) \to \mathrm{Sym}(\Phi^*\Theta_Y)$ *given by the differential of Φ.*

Proof Let $\mathcal{A}$ be the sheaf of $\mathcal{O}_X$-algebras which is the image of the map $\mathrm{Sym}(D\Phi) : \mathrm{Sym}(\Theta_X) \to \mathrm{Sym}(\Phi^*\Theta_Y)$ given by the differential of Φ. Then, since we have a natural surjective map $\mathrm{Sym}(\Theta_X) \twoheadrightarrow \mathcal{A}$, we can see that $\mathrm{Specan}(\mathcal{A})$ is defined as a closed complex-analytic subspace of $T^*X \cong \mathrm{Specan}\,\mathrm{Sym}(\Theta_X)$.

We claim first that the restriction of this subspace over U is exactly $T^*_\Phi U = T^*_\Phi X|_U$. To show this, we recall that a complex-analytic submersion of smooth manifolds is given locally by a coordinate projection — hence, by shrinking to small enough open $V \subseteq U$ around each point, we obtain $\Theta_V \cong \mathcal{O}_V^{\oplus n}$ and $\Phi^*\Theta_Y|_V \cong \mathcal{O}_V^{\oplus m}$ for n and m the respective dimensions of X and Y, with $D\Phi$ being given by the projection onto the last m coordinates. As such, $\mathrm{Sym}(D\Phi)$ is the already-surjective map $\mathcal{O}_V[\xi_1, \ldots, \xi_n] \twoheadrightarrow \mathcal{O}_V[\xi_1, \ldots, \xi_n]/(\xi_1, \ldots, \xi_{n-m}) \cong \mathcal{O}_V[\xi_{n-m+1}, \ldots, \xi_n]$ and so we can see that the corresponding map of cones is the closed inclusion $V \times \mathbb{C}^m \hookrightarrow V \times \mathbb{C}^n \cong T^*V$ of the sub-bundle which is zero locus of the first $n-m$ coordinates of $\mathbb{C}^n$. By considering a dual basis for the tangent bundle, we find that this is exactly the locus of covectors which vanish on the kernel of $D\Phi$, and it is already closed, so it is equal to $T^*_\Phi V$ as a subspace of T^*V. The local equalities at each point combine to give the claim over U.

Thus $\operatorname{Specan}(\mathcal{A})$ contains $\{(x,\eta) \in T^*U \mid \eta(\ker D\Phi|_x) = 0\}$; since it is closed, it therefore contains $T^*_\Phi X$ as well. It remains to show that this containment is an equality on the level of complex-analytic spaces. Let $\mathcal{I}$ be the ideal sheaf cutting out $T^*_\Phi X$ in $\operatorname{Specan}(\mathcal{A})$; by the invariance of both spaces involved under scaling, we can see that the support of $\mathcal{I}$ must meet the zero section of T^*X if it is non-empty and so it is enough to show equality locally at points of the zero section.

At each point of the zero section, however, the local rings of $\operatorname{Specan}\operatorname{Sym}(\Theta_X)$ and $\operatorname{Specan}\operatorname{Sym}(\Phi^*\Theta_X)$ are, if we set $n := \dim X$ and $m := \dim Y$, of the forms $\mathbb{C}\{x_1,\dots,x_n,\xi_1,\dots,\xi_n\}$ and $\mathbb{C}\{x_1,\dots,x_n,\xi'_1,\dots,\xi'_m\}$ respectively, and by Proposition 2.1.6 the local ring of $\operatorname{Specan}(\mathcal{A})$ at this point will be the image of the map of local rings ultimately induced by $D\Phi$. Since $\mathbb{C}\{x_1,\dots,x_n,\xi'_1,\dots,\xi'_m\}$ is an integral domain and so this is the quotient of $\mathbb{C}\{x_1,\dots,x_n,\xi_1,\dots,\xi_n\}$ by a prime ideal, this implies that $\operatorname{Specan}(\mathcal{A})$ is an irreducible complex-analytic space germ at the point in question. Then, if we apply Theorem III.C.16 of [GR65] using the closed subspace $A := \operatorname{Specan}(\mathcal{A}) \cap (T^*X|_{X\setminus U})$, we find that $\{(x,\eta) \in T^*U \mid \eta(\ker D\Phi|_x) = 0\}$ is locally dense in $\operatorname{Specan}(\mathcal{A})$, so its closure $T^*_\Phi X$ is locally equal to $\operatorname{Specan}(\mathcal{A})$ as sets and hence, since both spaces are locally reduced, as spaces. The result follows.

As a consequence, we can see immediately that our definition is independent of the chosen U.

This alternate description of the relative conormal space allows us to generalize Definition 3.2.3 beyond the situation where the spaces are smooth and the map is a submersion on a dense open subspace, to any map of complex-analytic spaces or, indeed, of schemes at least over a perfect field. However, since we will not actually have a use for relative conormal spaces in this level of generality, we will not develop the idea any further here.

We now note that, since the relative cotangent space is a closed subspace of the cotangent bundle, it comes endowed with some extra structure. In particular, per our discussion in Subsection 2.2.3, the cotangent bundle of a smooth space X arises as the analytic spectrum of the symmetric algebra generated by the sheaf of derivations Θ_X, but there is also a more informative way to endow Θ_X with a multiplication: From the definitions of Ω_X and Θ_X given in Definition 2.2.12, we can identify each derivation with the corresponding $\mathbb{C}$-endomorphism of $\mathcal{O}_X$ and use the composition of endomorphisms. Then, treating each element of $\mathcal{O}_X$ itself as an endomorphism of $\mathcal{O}_X$ by multiplication, we arrive at the following definition:

Definition 3.2.5 (e.g., [HTT08])**.** Let X be a smooth complex-analytic space. Then we define the **sheaf of differential operators** on X to be the sheaf $\mathcal{D}_X$ of (noncommutative) subalgebras of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X.

This sheaf comes with a standard filtration by orders of differential operators; taking the associated graded recovers the commutative algebra sheaf $\mathrm{Sym}(\Theta_X)$. Although we will not pursue it in any detail, we note that considering differential equations on X by way of quasicoherent sheaves of modules over the sheaf of differential operators gives rise to the theory of **$\mathcal{D}$-modules** — see, e.g., [HTT08].

Of particular interest to us is the observation that, since $\mathcal{D}_X$ is a filtered sheaf of noncommutative algebras whose associated graded is commutative, taking commutators of representatives induces a **Poisson bracket** on the associated graded in addition to the usual multiplication — details of the construction and definition can be found in [Gab81]. The bracket operation in turn induces a map $\mathcal{O}_{T^*X} \to \Theta_{T^*X}$ which itself satisfies the Leibniz rule and hence, by the universal property of Ω_{T^*X}, an $\mathcal{O}_{T^*X}$-module map $\Omega_{T^*X} \to \Theta_{T^*X}$. This map is an isomorphism, so we can regard its inverse as defining an element $\omega \in \Omega^2_{T^*X}(T^*X)$, which is a **symplectic form** and hence naturally endows T^*X with the structure of a **symplectic manifold** (e.g., Appendix E of [HTT08]).

In particular, we can now ask how subsets of the cotangent bundle, such as the relative conormal space, relate to the symplectic structure. In the case of functions to $\mathbb{C}$, we have the following result:

Theorem 3.2.6 (e.g., [GM18]). *Let X be an open subset of $\mathbb{C}^{n+1}$ and $f : X \to \mathbb{C}$ a non-constant holomorphic function. Then $T^*_f X|_{f^{-1}(0)}$ is set-theoretically a* **Lagrangian subspace** *of the restricted cotangent bundle $T^*X|_{f^{-1}(0)}$.*

The definition of a Lagrangian subspace may be found in, e.g., Appendix E of [HTT08] — for our purposes, the exact details will not be important. The proof of this result is by realizing $T^*_f X|_{f^{-1}(0)}$ as the characteristic variety of a sheaf of **vanishing cycles** on $f^{-1}(0)$; this is discussed from the perverse-sheaf-theoretic perspective in [GM18], and the original proof from the perspective of regular holonomic $\mathcal{D}$-modules can be found in [BMM94]. Since the characteristic varieties of holonomic $\mathcal{D}$-modules are Lagrangian by definition (e.g., [HTT08]), this gives the theorem as stated here.

For our purposes, the importance of this theorem will stem from the following result. The symplectic form ω on T^*X can be shown to arise as the exterior derivative of the **canonical 1-form** $\alpha \in \Omega_{T^*X}(T^*X)$ — see, e.g., Appendix E of [HTT08]. Then we have:

Proposition 3.2.7 (cf. Section E.3 of [HTT08]). *Let X be a smooth complex-analytic space and $\Lambda \subset T^*X$ a conic complex-analytic subset which is moreover Lagrangian. Then, for any complex-analytic submanifold $S \subseteq \Lambda$, the pullback of the canonical 1-form α to S is identically zero.*

Proof By Corollary E.3.2 of [HTT08], the pullback of α to the regular locus of Λ is zero. Then the result follows by Corollary 8.3.6(i) of [KS94] (applied to the holomorphic 1-form α by the usual identification of holomorphic forms with complexified real forms of type $(1, 0)$ — see, e.g., [Huy05]).

3.2.3 Thom Condition via the Relative Conormal Space

We can now restate the Thom condition of Definition 3.2.1 in terms of the relative conormal space of Definition 3.2.3 in cases of interest to us. We first recall the following well-known construction from differential geometry:

Definition 3.2.8. Let X be a smooth complex-analytic space and $Y \subseteq X$ a smooth locally closed complex analytic subspace, with $i : Y \hookrightarrow X$ the inclusion. Then we define the **conormal bundle** to Y in X as a closed subspace of the restricted cotangent bundle $T^*X|_Y$ by

$$T^*_Y X := \{(y, \eta) \in T^*X|_Y \mid \eta(T_y Y) = 0\},$$

considered as a complex-analytic space with the reduced structure.

That is, this is the bundle of covectors of X at points of Y vanishing on the tangent space of Y; we can see that this is indeed a vector bundle by considering local coordinates around each point of Y.

Then, under the right circumstances, we can reformulate the Thom condition as follows:

Proposition 3.2.9 (cf. [BMM94])**.** *Let $\Phi : X \to Y$ be a map between smooth complex-analytic spaces which is a submersion on an open dense subset of X. Then, for any locally closed complex-analytic submanifold S of the locus Σ where Φ drops rank such that $\Phi|_S$ has constant rank, the pair $(X \setminus \Sigma, S)$ satisfies the Thom (a_Φ)-condition if and only if, for each fiber M of $\Phi|_S$, $T^*_\Phi X|_M \subseteq T^*_M X$ set-theoretically.*

The stronger version of this statement implicit in the definition of the Thom condition given in [BMM94] uses a slightly more general definition of the relative conormal space than that of Definition 3.2.3, so for the sake of simplicity we will not concern ourselves with it. The proposition as stated follows from the definitions by observing that the set-theoretic condition $T^*_\Phi X|_M \subseteq T^*_M X$ is equivalent to the requirement that the covectors cutting out the limiting tangent spaces to fibers of $\Phi|_{X\setminus\Sigma}$ all vanish on TM.

We can now verify that Thom stratifications exist in nice enough circumstances; first we require the following simple lemma.

Lemma 3.2.10. *Let $f : X \to S$ be a nowhere-constant map of smooth complex-analytic spaces with S a curve. Then f is smooth of relative dimension $\dim X - 1$ and the Jacobian ideal $\mathcal{J}_f$ of Definition 3.1.4 is locally generated by the partial derivatives of f.*

Proof Since X is smooth, it is locally isomorphic as complex-analytic spaces to Euclidean space by Theorem 3.1.2, so it suffices to prove the statements locally on $\mathbb{C}^{n+1}$. At each point, the local ring can then be taken to be the convergent power series ring $\mathbb{C}\{x_0, \ldots, x_n\}$, which is in particular an integral domain; it then follows by the Principal Ideal Theorem (e.g., Theorem 12.3.3 of [Vak23]) and the fact that f is taken to be nowhere constant that the local fiber dimension is n. By Theorem 2.4.4, it then follows that f is flat of relative dimension n.

To show that the Jacobian ideal is as claimed, we take the **relative cotangent sequence** $f^*\Omega_S \to \Omega_X \to \Omega_f \to 0$, for which see, e.g., [Tei77; Eis04]. By the smoothness of X and S, this realizes Ω_f as the cokernel of a map of locally free sheaves — hence, by the definition of the nth Fitting ideal sheaf (e.g., [Tei77; Eis04]), $\mathcal{J}_f := \mathrm{Fitt}_n(\Omega_f)$ is locally given by the 1×1 minors of the corresponding matrix. This proves the result.

Now, using the method of proof of, e.g., Proposition 8.3.10 of [KS94], we have:

Theorem 3.2.11 (cf. [Hir77; BMM94])**.** *Let $f : X \to S$ be a nowhere-constant map of smooth complex-analytic spaces with S a curve such that $f(\Sigma_f)$ is a discrete set. Then f admits a complex-analytic Thom stratification.*

Proof We stratify S such that the strata are the points of $f(\Sigma_f)$ and the complement S_{amb} of the union of these points; it is not difficult to see that this stratification satisfies the Whitney conditions. Set $X_{\mathrm{amb}} := f^{-1}(S_{\mathrm{amb}})$ and, for each point $s \in f(\Sigma_f)$, consider the fiber $X_s := f^{-1}(s)$ over s.

By Theorem 3.1.13, the proper map $\mathbb{P}T^*_f X|_{X_s} \to X_s$ admits a complex-analytic Whitney stratification. We claim that, for each stratum M of X_s, $T^*_f X|_M \subseteq T^*_M X$ set-theoretically. To show this, we note first that it is enough to verify the assertion away from the zero section in $T^*X|_M$, since the complements of the zero section in $T^*_f X|_M$ and $T^*_M X$ are dense. Since $T^*_f X|_M$ is naturally a $\mathbb{C}^*$-bundle over $\mathbb{P}T^*_f X|_M$, we can see for each stratum of $\mathbb{P}T^*_f X|_{X_s}$ over M that its inverse image N in $T^*_f X|_M$ is a complex-analytic submanifold of $T^*_f X|_{X_s}$; Theorem 3.2.6 (applied locally) and Proposition 3.2.7 then imply that the pullback of the canonical 1-form to N is identically zero.

Around each point of M, let $x_1, \ldots, x_n$ be local coordinates for X, such that M is given by the vanishing of the coordinates $x_1, \ldots, x_k$. Then we can see that T^*X is locally a trivial vector bundle with coordinates $\xi_1, \ldots, \xi_n$, and the canonical 1-form is locally given by $\alpha = \xi_1 dx_1 + \ldots + \xi_n dx_n$ (e.g., Appendix E of [HTT08]). Since N is mapped into M by the projection, we can see that in our local patch the coordinates $x_1, \ldots, x_k$ vanish on N and so $dx_1, \ldots, dx_k$ do as well. Moreover, since N is mapped surjectively and submersively to M, the coordinates $x_{k+1}, \ldots, x_n$ of M are linearly independent coordinates on N and thus their differentials $dx_{k+1}, \ldots, dx_n$ are linearly independent as well. Hence $0 = \alpha|_M = \xi_{k+1} dx_{k+1} + \ldots + \xi_n dx_n$ implies that $\xi_i = 0$ for all $k < i \leq n$. Since x_i for $k < i \leq n$ are the coordinates of M, however, the vanishing of these ξ_i exactly defines $T^*_M X$ and so $N \subseteq T^*_M X$.

Applying this result across all strata of the projectivized relative conormal space verifies the claim. We can then take our Whitney stratification of X to be the one whose strata are X_{amb} and the chosen strata of X_s for each such s; this indeed gives a Whitney stratification for f since the Whitney conditions with respect to the ambient stratum are trivial and f takes strata of X surjectively and submersively to those of S by construction. Moreover, the Thom (a_f) condition for strata contained in a single fiber X_s reduces to the Whitney (a) condition, and all strata satisfy the Thom condition with respect to the ambient stratum by Proposition 3.2.9.

Note that the requirement that X be smooth can be circumvented — compare the results of [Hir77] and [BMM94].

3.3 Thom Condition via Flatness

We now turn to the situation which will concern us in Chapter 4 — a holomorphically-parameterized family of holomorphic functions on a smooth complex-analytic space. Specifically, we will build up to the proof of Theorem 3.3.3, which gives an algebro-geometric condition under which we can produce a stratification partially satisfying the Thom condition of Definition 3.2.1 for such a family.

3.3.1 Specialization of the Relative Conormal Space

As discussed in Subsection 3.2.3, the Thom condition is intimately related to the relative conormal space of Definition 3.2.3. We then want to know how relative conormal spaces behaves in families — in particular, how the relative conormal space of a function defining a family is affected by **specialization** to a particular choice of parameters. (For results on the specialization of the relative conormal space in various other settings, see [HMS84; LT88; HM96; GM18].)

Before stating our own result in this vein, we need the following general lemma, which says essentially that the behavior under pullback of a map between flat objects is controlled by that of its cokernel:

Lemma 3.3.1 ([Hof]). *Let R be a ring, $0 \to A_3 \to A_2 \to A_1 \to A_0 \to 0$ an exact sequence of R-modules such that A_1 and A_2 are flat over R, and B another R-module. Then the following hold:*

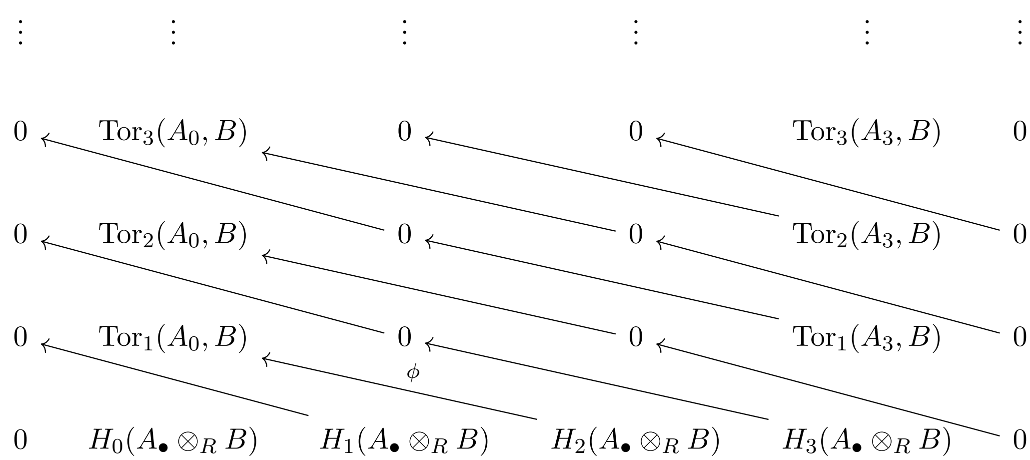
- $H_2(A_\bullet \otimes_R B) \cong \operatorname{Tor}_1^R(A_0, B)$.
- $H_3(A_\bullet \otimes_R B) \cong \operatorname{Tor}_2^R(A_0, B)$.
- $\operatorname{Tor}_i^R(A_3, B) \cong \operatorname{Tor}_{i+2}(A_0, B)$ *for all integers $i \geq 1$.*

Proof Let $\cdots \to F_2 \to F_1 \to F_0 \to B \to 0$ be a flat resolution of B. We can then form a first-quadrant double complex $A_\bullet \otimes_R F_\bullet$ whose (i, j)th entry is $A_i \otimes F_j$ using the usual sign trick (see, e.g., Section 2.7 of [Wei94]) and consider both of the corresponding spectral sequences (e.g., Section 5.6 of [Wei94]). By the flatness of the F_j and the exactness of $A_\bullet$, the rows of this double complex are exact, so the convergence of the spectral sequence given by the filtration by rows tells us that the homology of the associated total complex vanishes.

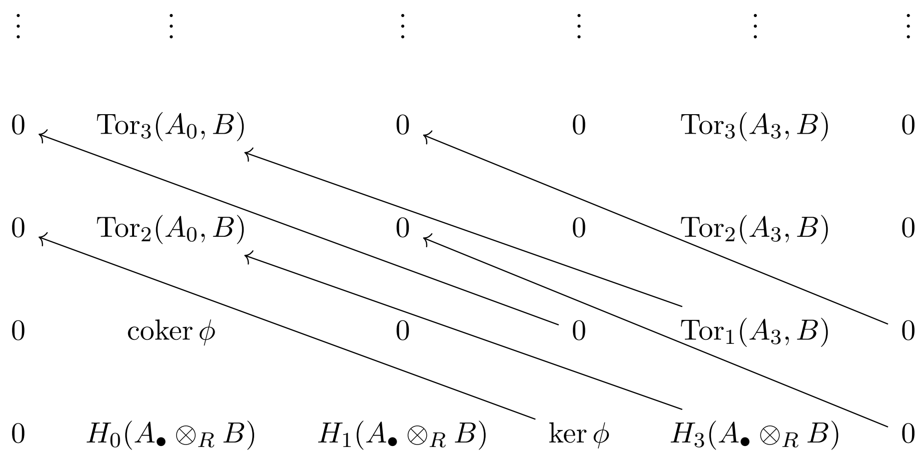
On the other hand, if we take the spectral sequence given by the filtration by columns, we find that the E^1 page has the following form:

$$
\begin{array}{cccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longleftarrow & \mathrm{Tor}_3(A_0, B) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \mathrm{Tor}_3(A_3, B) & \longleftarrow & 0 \\
0 & \longleftarrow & \mathrm{Tor}_2(A_0, B) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \mathrm{Tor}_2(A_3, B) & \longleftarrow & 0 \\
0 & \longleftarrow & \mathrm{Tor}_1(A_0, B) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \mathrm{Tor}_1(A_3, B) & \longleftarrow & 0 \\
0 & \longleftarrow & \mathrm{Tor}_0(A_0, B) & \longleftarrow & \mathrm{Tor}_0(A_1, B) & \longleftarrow & \mathrm{Tor}_0(A_2, B) & \longleftarrow & \mathrm{Tor}_0(A_3, B) & \longleftarrow & 0
\end{array}
$$

Hence we can see immediately that the E^2 page is as follows:



The E^3 page:



Since the maps on all higher-indexed pages must be zero, the convergence of the spectral sequence to the homology of the associated total complex, which we have already shown is zero, gives us the desired isomorphisms.

We have stated this lemma in the language of rings and modules for simplicity, but it will not be difficult to use it to obtain results on the level of spaces and sheaves.

We can now apply this to the question of the specialization of the relative conormal space. Observe that, for a smooth map, this space is a vector bundle over the domain — that is, the only interesting local behavior arises at the criticalities of the map, and hence we should expect such behavior to be controlled in some sense by the Jacobian ideal sheaf of Definition 3.1.4. Indeed, we have:

Lemma 3.3.2 ([Hof]). *Let X and U be smooth complex-analytic spaces. Let $\pi : X \times U \to U$ be the projection and suppose we have a map $F : X \times U \to \mathbb{C}$ of complex-analytic spaces such that F is nowhere constant on each fiber of π. Suppose that the closed embedding $\Sigma_{F\times\pi} \hookrightarrow X \times U$ is flat over U in the sense of Definition 2.3.14.*

Then, for each point $u \in U$, we have $T^*_{F\times\pi}(X\times U)|_{\pi^{-1}(u)} = T^*_{f_u}X \times_{\pi^{-1}(u)} \pi^*(T^*U)|_{\pi^{-1}(u)}$ as complex-analytic subspaces of the restricted cotangent bundle $T^*(X \times U)|_{\pi^{-1}(u)} = T^*X \times_{\pi^{-1}(u)} \pi^*(T^*U)|_{\pi^{-1}(u)}$, where $f_u := F|_{\pi^{-1}(u)}$.

Proof Let $i : X \cong \pi^{-1}(u) \hookrightarrow X \times U$ be the inclusion. Then the restricted cotangent bundle $T^*(X\times U)|_{\pi^{-1}(u)}$ on X is the vector bundle corresponding under the Fischer-Prill Theorem 2.2.5 to the coherent sheaf $i^*\Theta_{X\times U}\cong \Theta_X\oplus i^*\pi^*\Theta_U$.

Likewise, by Proposition 3.2.4, the restricted relative conormal space $T^*_{F\times\pi}(X\times U)|_{\pi^{-1}(u)}$ corresponds to the pullback along i of the image of the map $\mathrm{Sym}(D(F\times\pi)):\mathrm{Sym}(\Theta_{X\times U})\to\mathrm{Sym}((F\times\pi)^*\Theta_{\mathbb{C}\times U})$ of graded sheaves of algebras on $X\times U$; letting $\rho : X\times U\to X$ be the projection and noting that $\Theta_{X\times U}\cong\rho^*\Theta_X\oplus\pi^*\Theta_U$ and $(F\times\pi)^*\Theta_{\mathbb{C}\times U}\cong F^*\Theta_{\mathbb{C}}\oplus\pi^*\Theta_U$, we find that this can be written as a map $\mathrm{Sym}(\rho^*\Theta_X)\otimes\mathrm{Sym}(\pi^*\Theta_U)\to\mathrm{Sym}(F^*\Theta_{\mathbb{C}})\otimes\mathrm{Sym}(\pi^*\Theta_U)$. Hence we can study $\mathrm{Sym}(D(F\times\pi))$ in terms of the maps $\mathrm{Sym}(D(F\times\pi)|_{\rho^*\Theta_X}):\mathrm{Sym}(\rho^*\Theta_X)\to\mathrm{Sym}(F^*\Theta_{\mathbb{C}})\otimes\mathrm{Sym}(\pi^*\Theta_U)$ and $\mathrm{Sym}(D(F\times\pi)|_{\pi^*\Theta_U}):\mathrm{Sym}(\pi^*\Theta_U)\to\mathrm{Sym}(F^*\Theta_{\mathbb{C}})\otimes\mathrm{Sym}(\pi^*\Theta_U)$ on the two factors of the tensor product. Observe that the endomorphism of $\mathrm{Sym}(F^*\Theta_{\mathbb{C}})\otimes\mathrm{Sym}(\pi^*\Theta_U)$ induced by the natural map $\mathrm{Sym}(F^*\Theta_{\mathbb{C}})\to\mathrm{Sym}(F^*\Theta_{\mathbb{C}})\otimes\mathrm{Sym}(\pi^*\Theta_U)$ and $\mathrm{Sym}(D(F\times\pi)|_{\pi^*\Theta_U})$ is, in fact, an automorphism with inverse given by pre- and post-composing the same map with $\mathrm{id}_{\mathrm{Sym}(F^*\Theta_{\mathbb{C}})}\otimes(-1)$, since $\mathrm{Sym}(D(F\times\pi)|_{\pi^*\Theta_U})$ is induced by the direct product $DF|_{\pi^*\Theta_U}\times D\pi|_{\pi^*\Theta_U}=DF|_{\pi^*\Theta_U}\times\mathrm{id}_{\pi^*\Theta_U}$ of maps of coherent sheaves; denote this inverse by ψ. Since π is constant with respect to the coordinates of X, we can see moreover that $\mathrm{Sym}(D(F\times\pi)|_{\rho^*\Theta_X})$ is induced by tensoring the differential $\mathrm{Sym}(DF|_{\rho^*\Theta_X}):\mathrm{Sym}(\rho^*\Theta_X)\to\mathrm{Sym}(F^*\Theta_{\mathbb{C}})$ over $\mathcal{O}_{X\times U}$ with $\mathrm{Sym}(\pi^*\Theta_U)$.

As such, we find that $\psi\circ\mathrm{Sym}(D(F\times\pi)|_{\rho^*\Theta_X})=\mathrm{Sym}(D(F\times\pi)|_{\rho^*\Theta_X})$ and $\psi\circ\mathrm{Sym}(D(F\times\pi)|_{\pi^*\Theta_U})$ is the natural map $\mathrm{Sym}(\pi^*\Theta_U)\to\mathrm{Sym}(F^*\Theta_{\mathbb{C}})\otimes\mathrm{Sym}(\pi^*\Theta_U)$. Therefore, by our observation on $\mathrm{Sym}(D(F\times\pi)|_{\rho^*\Theta_X})$, $\psi\circ\mathrm{Sym}(D(F\times\pi))$ is the map given by $\mathrm{Sym}(DF|_{\rho^*\Theta_X})\otimes\mathrm{id}_{\mathrm{Sym}(\pi^*\Theta_U)}$. Since ψ is an automorphism, this means that in particular that these maps have the same kernel and hence their images are the same as quotients of $\mathrm{Sym}(\Theta_{X\times U})$. Therefore $T^*_{F\times\pi}(X\times U)|_{\pi^{-1}(u)}$ in fact corresponds to the pullback along i of the image of $\mathrm{Sym}(DF|_{\rho^*\Theta_X})\otimes\mathrm{id}_{\mathrm{Sym}(\pi^*\Theta_U)}:\mathrm{Sym}(\Theta_{X\times U})\to\mathrm{Sym}((F\times\pi)^*\Theta_{\mathbb{C}\times U})$. By the flatness of the locally free sheaf $\pi^*\Theta_U$ and the consequent flatness of $\mathrm{Sym}(\pi^*\Theta_U)$, this sheaf is given by tensoring the image of the map $\mathrm{Sym}(DF|_{\rho^*\Theta_X}) : \mathrm{Sym}(\rho^*\Theta_X) \to \mathrm{Sym}(F^*\Theta_{\mathbb{C}})$ itself by $\mathrm{Sym}(\pi^*\Theta_U)$ and then pulling back along i.

On the other hand, $i^* \mathrm{Sym}(DF|_{\rho^*\Theta_X}) = \mathrm{Sym}(Df_u)$, so the image of this map corresponds to the complex-analytic cone $T^*_{f_u}X$ in T^*X. Since $i^* \circ (- \otimes \mathrm{Sym}(\pi^*\Theta_U)) = (- \otimes \mathrm{Sym}(i^*\pi^*\Theta_U)) \circ i^*$, it therefore suffices to show that pullback along i commutes with taking the image of $\mathrm{Sym}(DF|_{\rho^*\Theta_X})$. That is, if we write $\phi := \mathrm{Sym}(DF|_{\rho^*\Theta_X})$, we need only verify that

$$0 \to i^* \ker\phi \to i^* \mathrm{Sym}(\rho^*\Theta_X) \xrightarrow{i^*\phi} i^* \mathrm{Sym}(F^*\Theta_{\mathbb{C}}) \to i^* \operatorname{coker}\phi \to 0$$

remains exact. Since exactness can be checked on stalks, i^* becomes a tensor product by the appropriate quotient of $\pi^{-1}\mathcal{O}_U$ on stalks, and the stalks of $\mathrm{Sym}(\rho^*\Theta_X)$ and $\mathrm{Sym}(F^*\Theta_{\mathbb{C}})$ are both flat over those of $\mathcal{O}_{X\times U}$ and hence those of $\mathcal{O}_U$, we can apply Lemma 3.3.1 to further reduce the problem to showing that the stalks of $\operatorname{coker}\phi$ are flat over those of $\mathcal{O}_U$.

Hence, letting $x \in X$ be any point and setting $p := (x, u)$, we take explicit local trivializations of the vector bundles $(\rho^*\Theta_X)_p$ and $(F^*\Theta_{\mathbb{C}})_p$, letting their coordinates be $\xi_0, \ldots, \xi_n$ (for $x_0, \ldots, x_n$ local coordinates of X) and τ respectively. Then we must show the flatness over $\mathcal{O}_{U,u}$ of the cokernel of the homogeneous map $\mathcal{O}_{X\times U,p}[\xi_0, \ldots, \xi_n] \to \mathcal{O}_{X\times U,p}[\tau]$ induced by DF — this is given by $\xi_i \mapsto \frac{\partial F}{\partial x_i}\tau$. As such, the cokernel is a graded module whose degree-d part is given by $(\mathcal{O}_{X\times U,p}/(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})^d)\tau^d$.

Now note that, since f_u is non-constant on X near x by hypothesis, its fiber over $f_u(x)$, and hence the fiber of $F \times \pi$ over $(f_u(x), u)$, must have dimension n (for $\dim X =: n+1$). Hence, by the smoothness of the complex-analytic spaces involved and Theorem 2.4.4, $F \times \pi$ is flat at p of relative dimension n, justifying our use of the critical locus in the statement of the result. Indeed, as in the proof of Lemma 3.2.10, $\mathcal{J}_{F\times\pi}$ is the nth Fitting ideal sheaf of $\Omega_{F\times\pi}$; by noting that $(F\times\pi)^*\Omega_{\mathbb{C}\times U} \to$

$\Omega_{X\times U} \to \Omega_{F\times\pi} \to 0$ locally realizes $\Omega_{F\times\pi}$ as the cokernel of a map of free sheaves and locally considering the rank-($\dim U + 1$) minors of the corresponding Jacobian matrix, we can see that $(\mathcal{J}_{F\times\pi})_p = (\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})$. As such, our cokernel above is isomorphic to the direct sum of the stalks at p of the structure sheaves of all of the infinitesimal neighborhoods of $\Sigma_{F\times\pi}$ — these are flat over $\mathcal{O}_{U,u}$ by hypothesis, proving the result.

(As a side consequence of our methods in this proof, we note that the relative conormal space in this case is exactly the analytic spectrum of the blowup algebra sheaf $B_{\mathcal{J}_{F\times\pi}}\mathcal{O}_{X\times U}$, and so the projectivized relative conormal space is simply the blowup of $X \times U$ at $\Sigma_{F\times\pi}$.)

Thus, for such a parameterized family, the flatness of the normal cone to the critical locus $\Sigma_{F\times\pi}$ is enough to guarantee that the relative conormal space behaves predictably under specialization.

3.3.2 A Stratification Theorem

We are now prepared to prove the promised stratification theorem for a family of holomorphic functions by imposing flatness conditions on the embedding of the relative critical locus in the family; compare this to the remarks of Hironaka on the relationship between the Thom condition and flatness in Section 5 of [Hir77].

Theorem 3.3.3 ([Hof]). *Let X and U be smooth complex-analytic spaces. Let $\pi : X\times U \to U$ be the projection and suppose we have a map $F : X \times U \to \mathbb{C}$ of complex-analytic spaces such that F is nowhere constant on each fiber of π.*

Then there exists a complex-analytic Whitney stratification of $X \times U$ such that the ambient stratum is $(X \times U) \setminus \Sigma_{F\times\pi}$, the non-flat locus of the embedding $\Sigma_{F\times\pi} \hookrightarrow X \times U$ (in the sense of Definition 2.3.14) over U is a closed union of strata, $F \times \pi$ has constant rank on each stratum, and the Thom ($a_{F\times\pi}$) condition with respect to the ambient stratum is satisfied on any stratum not contained in this non-flat locus.

Proof As in the proof of Lemma 3.3.2, working in local coordinates demonstrates that $F \times \pi$ is flat and $\Sigma_{F\times\pi}$ is precisely the locus where the map drops rank.

Let L be the non-flat locus of the embedding $\Sigma_{F\times\pi} \hookrightarrow X \times U$ over U in $\Sigma_{F\times\pi}$. We begin by arguing that L is a closed complex-analytic set. By definition, L is exactly the non-flat locus of the normal cone $C := C_{\Sigma_{F\times\pi}}(X \times U)$ over U in $\Sigma_{F\times\pi}$; consider the maps $C \xrightarrow{\rho} \Sigma_{F\times\pi} \xrightarrow{\pi|_{\Sigma_{F\times\pi}}} U$ and let L' be the non-flat locus of C over U in C, so that $L = \rho(L')$. Now, since the natural quotient map q makes the complement of the zero section in C a $\mathbb{C}^*$-bundle over $\mathbb{P}C$ (because the corresponding sheaf of algebras is generated in degree 1) and hence faithfully flat over it as well, we can see that L' is a union of orbits of the natural $\mathbb{C}^*$-action on C. L' is a closed complex-analytic subset of C by Theorem IV.9 of [Fri67], so this implies that in fact L' is a union of orbits of the natural $\mathbb{C}$-action — that is, any failure of flatness at a point away from the zero section will induce a failure of flatness at the corresponding point of the zero section. Hence L is exactly the intersection of L' with the zero section in C, proving the claim.

Now let $R := T^*_{F\times\pi}(X \times U)$ be the relative conormal space and consider the natural projection $\mathbb{P}R \to X \times U$. This map is proper and hence, by applying Theorem 3.1.13 with the condition $Q(S, s) =$ "locally around s, $F \times \pi$ has constant rank on S and S is either contained in or disjoint from each of L and $\Sigma_{F\times\pi}$", we obtain a complex-analytic Whitney stratification of it such that $F \times \pi$ has constant rank on every stratum in the induced stratification of $X \times U$ and L is a union of strata. Moreover, $\Sigma_{F\times\pi}$ is a union of strata; we claim that, by merging all other strata and their inverse image strata, we can take the ambient stratum to be $(X \times U) \setminus \Sigma_{F\times\pi}$.

We verify that this modification does not alter any of our claims. $F \times \pi$ has constant rank on $(X \times U) \setminus \Sigma_{F\times\pi}$ by definition, so this hypothesis is preserved. Moreover, the Whitney conditions with respect to an ambient stratum are trivial, so merging lower strata into the ambient stratum cannot affect them. Likewise, the closure of each stratum of $X \times U$ remains a complex-analytic subspace, so the result is still a complex-analytic stratification. Since $L \subseteq \Sigma_{F\times\pi}$, L remains a union of strata. Note that $\mathbb{P}R$ is an isomorphism over $(X \times U) \setminus \Sigma_{F\times\pi}$, since the relative conormal space R is a line bundle on this locus — therefore merging the strata in the source does not cause any failure of the restricted projection to be a surjective submersion, and the resulting partition of $\mathbb{P}R$ remains a complex-analytic Whitney stratification by the same reasoning used on $X \times U$.

This being done, we show that the resulting Whitney stratification of $X \times U$ fulfills the claims of the theorem statement. It remains to verify the statement about the Thom condition; to this end, consider a stratum S of $\Sigma_{F\times\pi} \setminus L$.

To begin with, we consider $\pi|_S$. Since $S \subseteq \Sigma_{F\times\pi}$ and π has constant rank on $X \times U$, we can see that $F \times \pi$ is constant with respect to X everywhere on S; that is, $(F \times \pi)|_S$ factors through π. In particular, since $F \times \pi$ has constant rank on S, π does as well, and so the fibers of $\pi|_S$ are manifolds. Therefore let $u \in U$ and set $M := \pi^{-1}(u) \cap S$; by Proposition 3.2.9, it is enough to verify for each such u that $R|_M \subseteq T^*_M(X \times U)$ set-theoretically.

Since M is by definition contained in a single fiber of π, which we will denote by $\pi^{-1}(u)$ or X, we find that $T^*_M(X \times U) = T^*_M X \times_M \pi^*(T^*U)|_M$ as a subspace of $T^*(X \times U)|_M = T^*X|_M \times_M \pi^*(T^*U)|_M$; that is, all covectors arising by pullback from U vanish on TM. Likewise, taking the function $f := F|_{\pi^{-1}(u)}$ on X and noting that M is disjoint from L, we find by Lemma 3.3.2 that $R|_M = T^*_f X|_M \times_M \pi^*(T^*U)|_M$.

Observe that, by our prior reasoning from the containment of S in $\Sigma_{F\times\pi}$, $M \subseteq \Sigma_f$; that is, $f|_M$ is constant. Observe moreover that, since the strata of $\mathbb{P}R$ are mapped to the corresponding strata of $X \times U$ surjectively and submersively, the constant-rank requirement implies restriction over u likewise yields a Whitney stratification of $\mathbb{P}R|_u \to X$. The proof now proceeds more or less in the manner of that of Theorem 3.2.11 — the only subtlety is that, instead of smooth locally-closed subsets of a conic Lagrangian subset of T^*X, we now work with smooth locally-closed subsets of the product of such a subset with $\mathbb{C}^{\dim U}$. However, taking the "canonical 1-form" only with respect to the coordinates of X, we find that the appropriate analogue to Proposition 3.2.7 holds by essentially the same proof. The result follows.

Chapter 4

The Milnor Fibration

In Chapters 2 and 3 we have seen various techniques for dealing with spaces and maps defined locally by holomorphic functions. We now apply these to our titular object of study: the **Milnor fibration** of a holomorphic function at a critical point. In brief, this construction captures the local behavior of the function's smooth fibers near the point in question — its study serves as an analogue of **Morse theory** or **Picard-Lefschetz theory** for arbitrary critical points.

The Milnor fibration has been worked on extensively by a variety of authors since its introduction in [Mil68]. Despite this, there is no known general method of computing even the homology of its fiber for a given holomorphic function, even if we restrict to fairly well-behaved classes of functions such as homogeneous polynomials. In [Hof], the author of this thesis proposes that the key to this problem should arise from the study of the function's critical locus as a complex-analytic space with the potentially non-reduced structure given by Definition 3.1.4 and proves a relative version of this principle for families of holomorphic functions — here we provide relevant background on the study of the Milnor fibration, review the proof of this result, and summarize some of its consequences. Specifically, in Section 4.1, we give the definition of the Milnor fibration, establish its existence, and mention the basic results motivating the complex-analytic-space-theoretic viewpoint. Section 4.2 provides background information and machinery for the behavior of the Milnor fibration in families, and in Section 4.3 we conclude by discussing the main theorem (Theorem 4.3.1) and some examples of its use.

As mentioned, the study of the Milnor fibration is a broad subject, and the results we review will be only a small selection — for more comprehensive introductions, see [Dim92; Max19; Sea19; LNS20]. For the relationship of the Milnor fibration with the machinery of **perverse sheaves** (and hence $\mathcal{D}$**-modules**) and the **nearby and vanishing cycles functors** in particular, see [Max19; Max20].

4.1 Definitions and First Results

Let X be a smooth complex-analytic space and $f : X \to \mathbb{C}$ a nowhere-constant holomorphic function. Then, as mentioned in Chapter 1, the local behavior of f locally around points away from Σ_f (which is well-defined by Lemma 3.2.10) is necessarily given by a trivial fibration of the sort depicted in Figure 1.1, which we reproduce here for easy reference:

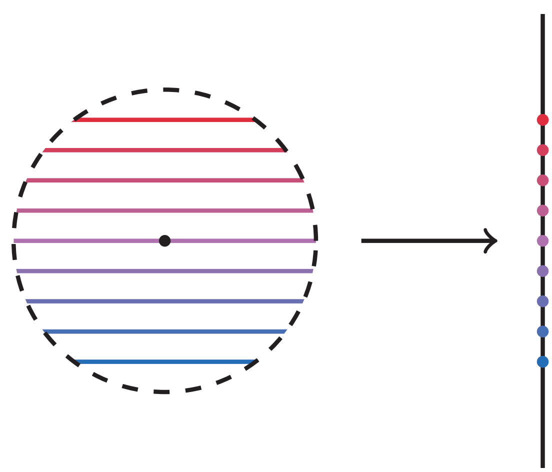


Figure 4.1: A reproduction of the local coordinate projection — e.g., $f(x, y) = y$ — depicted in Figure 1.1.

On the other hand, at points of Σ_f, the behavior of f is no longer quite so trivial. The fiber of f at such a point is singular; by working locally and discarding it, we obtain a fiber bundle, no longer trivial, which will be called the **Milnor fibration**. A simple example is depicted in Figure 1.2, which we again reproduce for easy reference:

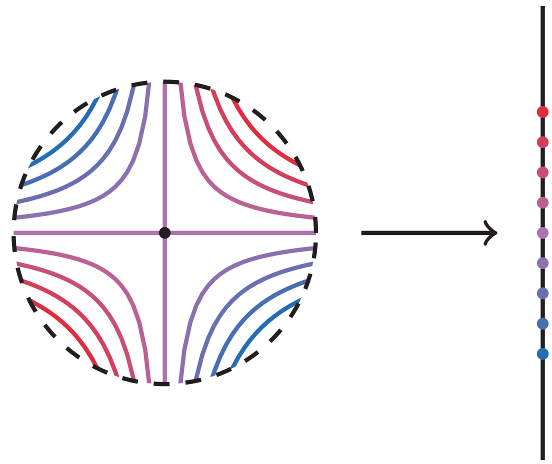


Figure 4.2: A reproduction of Figure 1.2, depicting local fibers around a critical point — e.g., the origin for $f(x, y) = xy$.

We now proceed as follows. Subsection 4.1.1 will introduce the definitions and results necessary to make the preceding discussion precise, while Subsection 4.1.2 will introduce globally-defined analogues to these in the special case of a homogeneous polynomial. Finally, Subsection 4.1.3 will detail some initial facts about the relationship between this fibration and the local structure of Σ_f which will eventually motivate our main results in Section 4.3.

4.1.1 The Local Milnor Fibration

Consider a **germ** of a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ — that is, an equivalence class of holomorphic functions $f : U \to \mathbb{C}$ on open sets $0 \in U \subseteq \mathbb{C}^{n+1}$ satisfying $f(0) = 0$ under the equivalence relation given by agreement on some such open subset. We will outline the construction of the Milnor fibration of f at the origin, following the proof of [Lê77]. We note first that the image of the critical locus of f is locally well-behaved:

Proposition 4.1.1. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ. Then, if we take a representative of f on a sufficiently small open neighborhood of 0 in $\mathbb{C}^{n+1}$, we have $f(\Sigma_f) \subseteq 0$ in $\mathbb{C}$.*

Proof Let $\varepsilon > 0$ be so small that f is defined in a neighborhood of $\bar{B}_\varepsilon$, for $B_\varepsilon \subset \mathbb{C}^{n+1}$ the open ball at the origin, and take the representative of f on such a neighborhood. Suppose toward a contradiction that we have a sequence x_i of points in $B_\varepsilon \cap \Sigma_f \cap f^{-1}(\mathbb{C}^*)$, where $\mathbb{C}^* := \mathbb{C}\backslash 0$; since $\bar{B}_\varepsilon \cap \Sigma_f$ is compact, we can pass to a subsequence to suppose that x_i converges to some point $x \in \bar{B}_\varepsilon \cap \Sigma_f$. Then, by the **Curve Selection Lemma**, proved by Milnor in the algebraic case in [Mil68] and more generally by Hironaka in [Hir73], there is a real-analytic curve $\rho : [0, r) \to \bar{B}_\varepsilon \cap \Sigma_f$ such that $\rho(0) = x$ and $\rho((0, r)) \subseteq B_\varepsilon \cap \Sigma_f \cap f^{-1}(\mathbb{C}^*)$. Since f is necessarily constant along any curve contained in Σ_f by Lemma 3.2.10, this yields the desired contradiction.

Hence 0, if it is contained in $f(\Sigma_f \cap B_\varepsilon)$ at all, is an isolated point of this set, and so the result holds for any neighborhood contained in the intersection of B_ε with the inverse image under f of a sufficiently small neighborhood of the origin in $\mathbb{C}$.

In light of this fact, the following result is now immediate from Theorem 3.2.11:

Proposition 4.1.2. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ. Then the representative of f on some sufficiently small open neighborhood of $0 \in \mathbb{C}^{n+1}$ admits a Thom stratification.*

Our proof of existence for the Milnor fibration will be by using this Thom stratification to control the smooth fibers of f. This will rely on the following well-known lemma:

Lemma 4.1.3. *Consider an open neighborhood U of the origin in $\mathbb{C}^{n+1}$ and fix a Whitney stratification of U. Then, for every $\varepsilon > 0$ in a sufficiently small neighborhood the origin in $\mathbb{R}_{\geq 0}$, the sphere S_ε of radius ε centered at the origin in $\mathbb{C}^{n+1}$ is transverse to all of the strata of our stratification.*

Proof Consider a stratum M not containing the origin and a point $x \in M$. Then, if we set $\varepsilon := |x|$, M fails to be transverse to S_ε at M if and only if $T_xM \subseteq T_xS_\varepsilon$; using the standard Riemannian metric on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, we can see that this will hold if and only if T_xM is orthogonal to the line ℓ from the origin to x. As such, if there were points of M arbitrarily close to the origin where this transversality failed, the Whitney condition (b) would not hold for M and the stratum containing the origin.

Then, since stratifications are by definition locally finite and the eventual transversality is immediate for the stratum containing the origin, the result follows.

We can now prove that the local smooth fibers of f over values near 0 fit together into a fiber bundle:

Theorem 4.1.4 ([Mil68; Lê77])**.** *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ. Then, for each $\varepsilon > 0$ in a sufficiently small open neighborhood of the origin in $\mathbb{R}_{\geq 0}$, we can choose small enough $\delta > 0$ such that f is defined on B_ε and the restriction*

$$f : B_\varepsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

is a diffeomorphically locally trivial fibration, where D_δ^* is the punctured open disk of radius δ at the origin in $\mathbb{C}$. Moreover, this is independent of the chosen δ and ε, and indeed of the chosen local coordinates.

Proof By Proposition 4.1.2 and Lemma 4.1.3, there exists an open neighborhood of the origin in $\mathbb{R}_{\geq 0}$ so small that, for any $\varepsilon > 0$ in it, f is defined and admits a Thom stratification on some neighborhood of $\bar{B}_\varepsilon$ and S_ε is transverse to all strata of this stratification.

Considering such an ε, we note for each point $x \in S_\varepsilon \cap f^{-1}(0)$ that, by the Thom condition and the openness of transversality, there is open neighborhood U_x of x in the domain of our representative of f such that the fibers of f at points of $U_x \cap f^{-1}(\mathbb{C}^*)$ are transverse to the spheres centered at the origin through those points. Therefore, we cannot have a sequence of points x_i of $S_\varepsilon \setminus \Sigma_f$ such that the fibers of f fail to be transverse to S_ε at the x_i and $f(x_i) \to 0$; if such a sequence were to exist, it would have a subsequence converging to some $x \in S_\varepsilon \cap f^{-1}(0)$ by the compactness of S_ε, which would then be eventually inside U_x, a contradiction.

Thus we can pick $\delta > 0$ so small that the fibers of $f|_{D^*_\delta}$ are transverse to S_ε and consider the restriction of f to $\bar{B}_\varepsilon \cap f^{-1}(D^*_\delta)$. This map is proper by the compactness of $\bar{B}_\varepsilon$ and, if we stratify its domain by the subsets $B_\varepsilon \cap f^{-1}(D^*_\delta)$ and $S_\varepsilon \cap f^{-1}(D^*_\delta)$, we can see that its restriction to each stratum is a smooth submersion, in the former case because the domain is disjoint from Σ_f and in the latter case exactly by the requirement that the fibers of f be transverse to S_ε. Since the Whitney conditions are trivial for the chosen strata, Thom's First Isotopy Lemma 3.1.14 then implies that this map is a homeomorphically trivial fibration whose trivializations are moreover diffeomorphic along each stratum. This proves the main claim.

The claimed independence of δ for a fixed ε follows by noting (e.g., Lemma 6.2.16 of [LNS20]) that such a locally trivial fibration over D^*_δ depends only on its restriction over any circle of radius less than δ centered at the origin. To prove independence of ε, we can modify our approach above to replace the compact set S_ε by the compact set $\bar{B}_{\varepsilon''} \setminus B_{\varepsilon'}$ for $0 < \varepsilon' < \varepsilon < \varepsilon''$ (with ε'' still sufficiently small) to choose a $\delta > 0$ such that the fibers of $f|_{D^*_\delta}$ are transverse to the spheres of radii between ε' and ε''; we can then apply Thom's First Isotopy Lemma to the restriction of $f \times \mathrm{id}_{(\varepsilon',\varepsilon'')}$ to the intersection of $f^{-1}(D^*_\delta) \times (\varepsilon', \varepsilon'')$ with the subspace of $\bar{B}_{\varepsilon''} \times (\varepsilon', \varepsilon'')$ whose fiber over each $r \in (\varepsilon', \varepsilon'')$ is $\bar{B}_r$. It follows by again applying Lemma 6.2.16 of [LNS20] that shrinking ε does not affect the fibration.

The proof of independence of local coordinates in the case of an isolated critical point can be found in, e.g., [LNS20]. The proof in general is similar to the proof of independence of ε, using a family of coordinate systems parameterized over the unit interval — we can then use a similar compactness argument to the above, with the Whitney condition (a) replacing the Thom condition (a_f), to show that there exists an open neighborhood of the origin in $\mathbb{R}_{\geq 0}$ small enough that, for $\varepsilon > 0$ contained in it, all of the boundary spheres $S^{(t)}_\varepsilon$ for the varying coordinate systems parameterized by $t \in [0, 1]$ are transverse to the Thom stratification. (Strictly speaking, we must here use a Nash modification of each stratum closure individually to account for behavior at the origin, but we omit the details.) We can then apply the compactness argument together with Thom (a_f) again to show that there is δ small enough that Thom's First Isotopy Lemma applies to the restriction of $f \times \mathrm{id}_{[0,1]}$ to the intersection of $f^{-1}(D_\delta^*) \times [0,1]$ with $\{(p,t) \in \mathbb{C}^{n+1} \times [0,1] \mid p \in \bar{B}_\varepsilon^{(t)}\}$, so that the result again follows by Lemma 6.2.16 of [LNS20].

We now make the following definitions:

Definition 4.1.5 ([Mil68; Lê77])**.** Let $f : X \to \mathbb{C}$ be a nowhere-constant holomorphic function on a smooth complex-analytic space. Considering the germ of f at a point $x \in X$, we call the fibration of Theorem 4.1.4 the **Milnor fibration** of f at x. We call its fiber the **Milnor fiber** of f at x and denote it by $\mathbb{F}_{f,x}$ — if the point x is clear from context, we drop it from the notations and terminology. We call the isotopy class of the diffeomorphism from $\mathbb{F}_f$ to itself induced by transport around a loop around the origin in the punctured disk (which is well-defined — see [LNS20]) the **Milnor monodromy**.

For a non-constant germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and a given coordinate system on $\mathbb{C}^{n+1}$, we say that a choice of $\varepsilon > 0$ small enough to define the Milnor fibration is a **Milnor radius** for f at the origin.

Note that the original construction in [Mil68] not quite the same — in the parlance, what we have defined is the **Milnor tube fibration**, whereas Milnor's original definition was of the **Milnor sphere fibration**. Since the two are equivalent (e.g., [Dim92]), we will suppress this distinction and work only with the fibration constructed in Theorem 4.1.4.

We note also the following well-known result:

Proposition 4.1.6. *Let $f : X \to \mathbb{C}$ be a nowhere-constant holomorphic function on a smooth complex-analytic space and suppose we are given a complex-analytic Whitney stratification of f. Then the diffeomorphism type of the local Milnor fibrations of f at points along each stratum is constant.*

We omit the proof, which uses **Thom's Second Isotopy Lemma**, the relative version of Lemma 3.1.14 mentioned in Subsection 3.2.1.

4.1.2 The Global Milnor Fibration of a Homogeneous Polynomial

In the special case of a homogeneous polynomial, we have another way of producing a fiber bundle by discarding the singular fiber:

Proposition 4.1.7 (e.g., [Dim92])**.** *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a nonzero homogeneous polynomial of degree d. Then the restriction $f : f^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$ defines a holomorphically locally trivial fibration, the fiber of which is a d-fold covering space of the complement of the hypersurface cut out by f in $\mathbb{P}^n$.*

The proof of this is by observing that $f(tx_0, \ldots, tx_n) = t^d f(x_0, \ldots, x_n)$ and using local branches of $\sqrt[d]{-}$ to obtain the local trivializations; taking the quotient of $f^{-1}(1)$ by the action of the dth roots of unity yields the claimed covering map.

Definition 4.1.8 (e.g., [Dim92])**.** Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a nonzero homogeneous polynomial. Then we call the restriction $f : f^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$ the **global** or **affine Milnor fibration** of the homogeneous polynomial f.

This terminology is justified by the following observation:

Proposition 4.1.9 (e.g., [Dim92])**.** *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a nonzero homogeneous polynomial. Then the affine Milnor fibration of f is diffeomorphically equivalent to the Milnor fibration of f at the origin.*

The proof is essentially by the fact that f behaves well under scaling of $\mathbb{C}^{n+1}$ by a positive real factor, which also gives us the following well-known result:

Proposition 4.1.10. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a nonzero homogeneous polynomial. Then every positive real number is a Milnor radius for f at the origin.*

The proof is by considering a complex-analytic Thom stratification of f such that the origin is its own stratum and every stratum is a conic subset of $\mathbb{C}^{n+1}$ — this can be obtained, for example, by taking the stratification induced by a Whitney stratification of the hypersurface cut out by f in $\mathbb{P}^n$ and using Theorem 4.2.1 of [BMM94].

Analogues of these results hold more broadly for **weighted-homogeneous** polynomials on $\mathbb{C}^{n+1}$ for any choice of weights, but we will not pursue the matter here.

4.1.3 First Relations with the Critical Locus

As discussed, holomorphic functions are already locally trivial at non-critical points — in particular, at such a point the Milnor fibration in fact extends over the origin in $\mathbb{C}$ to a trivial fibration over the disk with fiber an open ball of the appropriate dimension (Figure 4.1). Hence, at least in a crude way, the Milnor fibration of a function f at a point depends on the local structure of the critical locus Σ_f there. The following theorem of Milnor (in the algebraic case, and Hamm, in the holomorphic case) for isolated singularities beautifully refines this notion:

Theorem 4.1.11 ([Mil68; Ham71]). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ, and suppose that 0 is an isolated point of Σ_f. Then the Milnor fiber of f at the origin is homotopy-equivalent to a wedge sum of μ_f n-spheres S^n, where $\mu_f := \dim_{\mathbb{C}} \mathcal{O}_{\Sigma_f,0}$.*

Definition 4.1.12. The integer μ_f of Theorem 4.1.11 is called the **Milnor number** of f at the origin. In situations where the point in question must be specified, we will also denote this by $\mu_{f,0}$.

Here μ_f is finite precisely because the origin is an isolated point of Σ_f; hence $\mathcal{O}_{\Sigma_f,0}$ is Artinian and the dimension over $\mathbb{C}$ is simply its **length**. Note, crucially, that this result and definition depend on the structure of Σ_f as a complex-analytic space, which is to say of the **Jacobian ideal** $J_f := (\mathcal{J}_f)_0$ in the local ring $\mathcal{O}_{\mathbb{C}^{n+1},0}$ at the origin; as a set, Σ_f is simply a point, by hypothesis.

In the simplest case, we have the following widely-used terminology:

Definition 4.1.13. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ, and suppose that 0 is an isolated point of Σ_f. Then, if $\mu_f = 1$, we say that the origin is a **non-degenerate critical point**, **quadratic singularity**, or **Morse point** of f.

It is not difficult to see that in this case we can, up to a change of coordinates, take f to be the sum of the squares of all coordinate functions, and so Theorem 4.1.11 can be verified explicitly. Indeed, in this case the result was known before Milnor's work, forming the basis of **Picard-Lefschetz theory**, a complex-analytic analogue of **Morse theory** — see, e.g., [AGV88] and [GM88] respectively.

Theorem 4.1.11 naturally leads us to wonder what the local structure of Σ_f as a complex-analytic space can tell us about the Milnor fiber in the case of a non-isolated critical point of f. This will be the inspiration for our main results of Section 4.3. For now, we content ourselves with the following result:

Theorem 4.1.14 ([KM75])**.** *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ and let $s := \dim \Sigma_f$ be the dimension of the critical locus as a complex-analytic space germ at the origin. Then the Milnor fiber $\mathbb{F}_f$ is at least $(n-s-1)$-connected.*

Since the Milnor fiber of f has the homotopy type of a real cell complex of dimension at most n (see, e.g., Section 1.5 of the Introduction of [GM88]), this has the following immediate consequence:

Corollary 4.1.15. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ and let $s := \dim \Sigma_f$ be the dimension of the critical locus as a complex-analytic space germ at the origin. Then the reduced integral homology of the Milnor fiber satisfies $\tilde{H}_i(\mathbb{F}_f) = 0$ for all $i \notin [n-s, n]$.*

4.2 Background on the Milnor Fibration in Families

Our results in Section 4.3 will be relative in nature — that is, they will concern the behavior of the Milnor fibration under deformations of the function being considered. As we can surmise by starting with any function with a critical point at the origin and pre-composing it with a family of translations of the domain moving the origin off the critical locus, we cannot expect deformations to preserve the Milnor fibration in general. However, as Proposition 4.1.6, it is not entirely out of the question to find conditions under which perturbing the function preserves the critical locus.

Here we will review a selection of such conditions and related results from the literature — in keeping with our goal of understanding of the Milnor fibration using the machinery of complex-analytic spaces, we will consider only holomorphic deformations, although some authors which are only continuous or satisfy some weaker constraint than holomorphicity. In particular, we will consider the following situation:

Definition 4.2.1. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ. A **germ of a deformation** of f is defined to be the pair (F, π) for $F : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}, 0)$ a holomorphic function germ and $\pi : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}^u, 0)$ the projection onto the last u coordinates such that the restriction of F to $\pi^{-1}(0)$ is our original germ f.

If $F : X \times U \to \mathbb{C}$ is a holomorphic function for X a smooth complex-analytic spaces and U a neighborhood of the origin in $\mathbb{C}^u$, $\pi : X \times U \to U$ is the projection, and F is nowhere constant on the fibers of π, we say that the pair (F, π) is a **deformation** of F's restriction to each fiber of $\pi^{-1}(0)$ if its germ at each point of $X \times U$ is a deformation.

4.2.1 Tame Deformations and Splitting Techniques

As is often the case for local triviality statements in the study of singularities, many of our results on the consistency of the Milnor fibration under deformation depend on Thom's Isotopy Lemmas. The main challenge in proving such results from this perspective will then, as in Subsection 4.1.1, be to ensure the transversality of smooth fibers to appropriately-chosen boundary spheres. To keep track of such failures, we can consider the following object:

Definition 4.2.2 ([JST])**.** Let U be a smooth complex-analytic space, $\tilde{X}$ an open neighborhood of 0×0 in $\mathbb{C}^{n+1} \times U$, and $\pi : \tilde{X} \to U$ the projection. Let $F : \tilde{X} \to \mathbb{C}$ be a holomorphic function nowhere constant on the fibers of π and, fixing a coordinate system on $\mathbb{C}^{n+1}$, let $\rho : \mathbb{C}^{n+1} \to \mathbb{R}_{\geq 0}$ be the squared Euclidean distance from the origin. Letting $\Sigma_{F\times\pi\times\rho}$ denote the locus where this function's real Jacobian matrix drops rank, we define the **Milnor set** of $F \times \pi$ to be the closed subset

$$M(F \times \pi) := \overline{\Sigma_{F\times\pi\times\rho} \setminus (F \times \pi)^{-1}((F \times \pi)(\Sigma_{F\times\pi}))}$$

of $\tilde{X}$.

Thinking of U as the parameter space of the family of holomorphic functions defined by F and π, we see that this is exactly the closure of the locus where the smooth fibers of functions in the family fail to be transverse to the corresponding boundary spheres — hence controlling the Milnor set will be enough to give us the kind of consistency results we seek. Specifically, we can define:

Definition 4.2.3 ([JST])**.** Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ and (F, π) a germ of a deformation of f. Then we say that this deformation is **tame** if F has a representative on an open subset of $\mathbb{C}^{n+1+u}$ such that $M(F \times \pi) \cap \pi^{-1}(0) \cap \Sigma_{F\times\pi} \subseteq 0$.

This is to say that none of the critical points of f (since $\pi^{-1}(0) \cap \Sigma_{F\times\pi}$ is equal to Σ_f), except possibly the origin, are approached by sequences of points where the smooth fibers of $F \times \pi$ fail to be transverse to the corresponding spheres centered at the origin. In these circumstance, our deformation preserves the smooth fiber in the following sense:

Theorem 4.2.4 ([JST]; cf. [Hof])**.** *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ and (F, π) a germ of a tame deformation of f.*

Then, for each $\varepsilon > 0$ in a sufficiently small open neighborhood of the origin in $\mathbb{R}_{\geq 0}$, we can choose small enough $\delta > 0$ and $\gamma > 0$ such that F is defined on $B_\varepsilon \times B_\gamma$ and, if we let $\Delta := (F \times \pi)(\Sigma_{F\times\pi})$ be the image of the representative of the critical locus on this *open set, the restriction*

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap (F \times \pi)^{-1}((D_\delta \times B_\gamma) \setminus \bar{\Delta}) \to (D_\delta \times B_\gamma) \setminus \bar{\Delta}$$

is a diffeomorphically locally trivial fibration. Here $B_\varepsilon \subset \mathbb{C}^{n+1}$*,* $D_\delta \subset \mathbb{C}$*, and* $B_\gamma \subset \mathbb{C}^u$ *are the open balls of the specified radii.*

The proof is by applying Thom's First Isotopy Lemma 3.1.14 on $\bar{B}_\varepsilon \times B_\gamma$, analogously to the proof of Theorem 4.1.4. Note that the statement of this theorem differs slightly from the one in [JST], insofar as we allow more parameters and do not here claim the fibration is independent of the choices made. In the case of a one-parameter family, as in [JST], such independence can be established since Δ is well-defined as a germ of a closed complex-analytic curve (Proposition 2.4 of [JST]); in the general setting, this is not known a priori and so Δ must be defined with reference to the chosen representative for F.

Since the statement of the result is somewhat technical, let us take a moment to understand its main application. We can see that, if ε is sufficiently small, the restriction of this fibration over $(D_\delta \times 0) \setminus \bar{\Delta} = D_\delta^*$ (this equality can be verified by restricting over every curve and applying Proposition 2.4 of [JST], by the Curve Selection Lemma mentioned in Subsection 4.1.1) is simply the Milnor fibration of f at the origin. Hence the fiber of the fibration guaranteed by the theorem is $\mathbb{F}_f$ and so perturbing f by changing the parameters slightly still gives us information about the Milnor fiber of f — indeed, we can study the monodromy through perturbation in the family as well by moving a loop appropriately in $(D_\delta \times B_\gamma) \setminus \bar{\Delta}$.

Observe, however, that some care must be taken — as noted in [JST], it is a well-known issue in the study of families of holomorphic functions that the consistency of the Milnor radius is not guaranteed. That is, although the restriction of the fibration in Theorem 4.2.4 over a parameter value other than 0 has fibers given by the corresponding function in the deformation, it need not be the case that ε remains a valid Milnor radius for the deformed function, and so the theorem does not guarantee the constancy of the actual Milnor fibers in the family, merely that of the "semi-local smooth fibers" defined by the radius ε.

Although this is in some respects a disadvantage, since it limits the utility of Theorem 4.2.4 in showing that different functions have the same Milnor fiber, it also grants us a powerful tool: the ability to split the critical locus into simpler pieces and compute the contributions of each to the homology of the Milnor fiber separately. To be precise, we have the following result:

Theorem 4.2.5 ([Sie87]). *In the situation of Theorem 4.2.4, consider $t \in B_\gamma$ and set $f_t := F|_{B_\varepsilon \times t}$. Suppose that $\bar{\Delta} \cap (D_\delta \times t)$ is a finite set of points $q_1, \ldots, q_r$ and take $\mathbb{D}_1, \ldots, \mathbb{D}_r$ disjoint open disks around the respective q_i. Let $v_i \neq q_i$ be points of $\mathbb{D}_i$ for all i and let v be a point of $D_\delta \times t$ disjoint from all $\mathbb{D}_i$.*

Then, if we set $\mathbb{E} := {f_t}^{-1}(D_\delta \times t)$, $\mathbb{F} := {f_t}^{-1}(v)$, $\mathbb{E}_i := {f_t}^{-1}(\mathbb{D}_i)$ for all $1 \leq i \leq r$, and $\mathbb{F}_i := {f_t}^{-1}(v_i)$ for all $1 \leq i \leq r$, we have

$$\tilde{H}_k(\mathbb{F}_f) \cong H_{k+1}(\mathbb{E}, \mathbb{F}) \cong \bigoplus_{i=1}^{r} H_{k+1}(\mathbb{E}_i, \mathbb{F}_i)$$

for all integers k and f the original function being deformed.

Hence we can assess the homological contributions of each critical value of the deformed function separately — for examples of the use of this theorem, see [ST17] and Example 9.2 of [Hof], which we will revisit as Example 4.3.3.

4.2.2 Morsification and Finite Determinacy

The tame deformations of Definition 4.2.3 are useful essentially because of their good behavior near the boundary of a small sphere around the origin over the parameter value 0. One very natural situation where this occurs is the case of isolated singularities:

Proposition 4.2.6. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ, and suppose that 0 is an isolated point of Σ_f. Then every deformation of f is tame.*

This is immediate since in this case Σ_f is already set-theoretically contained in the origin. It then becomes profitable to consider the following notion:

Definition 4.2.7 (e.g., [AGV88])**.** Let X a smooth complex-analytic space and $f : X \to \mathbb{C}$ a holomorphic function with only isolated critical points. Then, if (F, π) is a deformation of f such that F's restriction over any parameter value other than 0 has only Morse critical points (Definition 4.1.13), we say that (F, π) is a **Morsification** of f.

Such deformations are easy to construct in general by the density of Morse functions (e.g., [GM88]) — the key point here is that the tameness of deformations of functions with isolated critical points guarantees that a Morsification of such a function will locally preserve smooth fibers by, e.g, Theorem 4.2.4 and so allow us to understand the original function in terms of the Picard-Lefschetz theory of the functions we deform to.

For functions with non-isolated criticalities, such deformations will not be tame in general, if at all, and hence are much less useful in the study of the Milnor fibration. However, there turns out to be a class of deformations which usefully split off Morse points from the critical locus of the function germ being deformed: **unfoldings** of the germ through an ideal with respect to which it has **finite extended codimension**. We do not give the exact definitions of these terms, which can be found in, e.g., [Bob04] — the key point, for which see [Bob04], or more explicitly the proof of Proposition 8.1 in [Hof], is that such an unfolding is holomorphically trivial locally at points away from the origin. Hence we arrive at following proposition, which together with Theorem 4.2.4 gives essentially the result of Theorem 2.2 of [Bob04]:

Proposition 4.2.8. *An unfolding of a function germ through an ideal with respect to which it has finite extended codimension is tame.*

This will follow from Theorem 4.3.1, since such unfoldings satisfy its hypotheses, as we will discuss in Subsection 4.3.1.

4.2.3 Lê Numbers

We now recall an entirely different theory of deformations for the Milnor fibration, the foundation of which lies in results on the behavior of the Milnor fibration upon restriction to a hyperplane through the origin. These are based on the following construction:

Definition 4.2.9 ([Lê73]). Let X be an open neighborhood of the origin in $\mathbb{C}^{n+1}$ and $f : X \to \mathbb{C}$ a holomorphic function. For a linear form $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$, set C_ℓ to be the locus where the differential of $f \times \ell$ drops rank.

Then the **polar locus** of f relative to the direction defined by ℓ is the closed set in X given by $\Gamma_{f,\ell} := \overline{C_\ell \setminus \Sigma_f}$.

This captures the failure of the family of hypersurfaces defined by ℓ to be transverse to the smooth points of fibers of f, much like the Milnor set of Definition 4.2.2. By a result of Lê, we can then use this to control the relationship between the Milnor fibration of f and that of the restriction of f to the hyperplane cut out by ℓ:

Theorem 4.2.10 ([Lê73]). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ and $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$ a generic linear form. Then $\Gamma_{f,\ell}$ is a well-defined complex-analytic curve germ at the origin and, if we let H be the hyperplane cut out by the vanishing of ℓ, the Milnor fiber $\mathbb{F}_f$ is homotopy-equivalent to a space obtained by attaching to $\mathbb{F}_{f|_H}$ a number of n-cells equal to the* **intersection number** *(see [Ful98]) at the origin of H and the hypersurface cut out by the vanishing of the germ f.*

As noted in [Mas95], this implies Theorem 4.1.14. Unfortunately, Lê's methods, which are Morse-theoretic, do not give insight into the nature of the attaching map, so iterative application of this result is not sufficient to understand the Milnor fiber of f completely. However, such an approach can be used to establish strong relative results on Milnor fiber consistency in deformations. Specifically, we can use the **Lê numbers** (e.g., [Mas95]) — quantities capturing the relative polar information which arises from iteratively slicing by the coordinate hyperplanes of a given coordinate system — to obtain the following result:

Theorem 4.2.11 ([Mas95]). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a non-constant holomorphic function and $s := \dim \Sigma_f$ the dimension of the critical locus as a complex-analytic space germ at the origin. Then, for a given deformation of f and a coordinate system on $\mathbb{C}^{n+1}$ subject to certain effective genericity conditions with respect to the deformation, the* *constancy of the Lê numbers at the origin under the deformation implies the constancy under the deformation of the homology of the Milnor fiber at the origin. If* $s \leq n-2$*, the homotopy type of the Milnor fibration is constant under the deformation; if, moreover,* $s \leq n-3$*, the same is true of the diffeomorphism type.*

This result shows that the strengths and weaknesses of the approach of controlling deformations through Lê numbers are more or less opposite to those of tameness-based methods; constancy of Lê numbers guarantees consistency of Milnor fibers themselves, not just semi-local smooth fibers, but by this very fact does not allow for splitting techniques in the style of Theorem 4.2.5.

4.3 Results Based on Flatness

We now turn to our main results, those which impose control over Milnor fibration's behavior under deformation through algebraic means. The results of Subsection 4.1.3 suggest that the topology of the Milnor fiber of a germ f should be recoverable through analysis of the structure of the critical locus as a complex-analytic space germ. In particular, we should expect a deformation (F, π) of f to give us useful information about the Milnor fibration so long as it also defines a deformation of the critical locus over the parameter space — since flatness is, as we have noted in Chapter 2, the algebro-geometric and complex-analytic notion of what it means for a map to be a deformation, and the critical locus $\Sigma_{F\times\pi}$ of any representative of the deformation gives the critical loci of the functions in the family on restriction to the corresponding fibers of π, we hope to find results in terms of the flatness of $\Sigma_{F\times\pi}$ over the parameter space.

In Subsection 4.3.1, we will see that this is true in an embedded sense — that is, if the embedding of the critical locus into the ambient total space of the deformation is flat, the deformation will be tame. Subsections 4.3.2 and 4.3.3 will explore consequences of this in well-behaved special cases.

4.3.1 The Main Theorem

As mentioned, we will here establish that the flatness of the embedding of the critical locus is enough to guarantee that a deformation is tame. Indeed, a stronger but more technical result will be true — as in the definition of tameness itself, we need only constrain behavior away from the origin on the special fiber of the deformation.

Specifically, recalling our definitions of a deformation germ (F, π) (Definition 4.2.1), the critical locus $\Sigma_{F\times\pi}$ of $F \times \pi$ (Definition 3.1.4), the flatness of a locally closed embedding (Definition 2.3.14), and the tameness of a deformation (Definition 4.2.3), we have:

Theorem 4.3.1 ([Hof])**.** *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ and $(F : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}, 0), \pi : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}^u, 0))$ a germ of a deformation of f. Suppose that the intersection with $\pi^{-1}(0)$ of the germ at the origin of the non-flat locus of the embedding $\Sigma_{F\times\pi} \hookrightarrow \mathbb{C}^{n+1+u}$ over $\mathbb{C}^u$ is set-theoretically contained in the origin. Then (F, π) is a tame deformation of f.*

Proof We take a representative of F and apply Theorem 3.3.3 to obtain a complex-analytic Whitney stratification of the neighborhood of the origin in $\mathbb{C}^{n+1+u}$ on which it is defined such that strata not contained in the non-flat locus satisfy the Thom condition with respect to the ambient stratum. By the reasoning of Lemma 4.1.3, the spaces $S_\varepsilon \times \mathbb{C}^u$ will be transverse to the strata at points of $\pi^{-1}(0)$ sufficiently close to the origin. Hence, as in the proof of Theorem 4.1.4, the Thom condition then lets us produce neighborhoods of each point of $\Sigma_{F\times\pi} \cap \pi^{-1}(0) = \Sigma_f \times 0$ outside the non-flat locus which are disjoint from the Milnor set $M(F \times \pi)$. The result follows.

We state the weaker but more legible version of this result independently:

Corollary 4.3.2. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ and $(F : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}, 0), \pi : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}^u, 0))$ a germ of a deformation of f. Suppose the germ at the origin of the embedding $\Sigma_{F\times\pi} \hookrightarrow \mathbb{C}^{n+1+u}$ is flat over $\mathbb{C}^u$. Then (F, π) is a tame deformation of f.*

Note by Theorem 4.2.4 that deformations satisfying these properties can then be used to study the Milnor fibration. As an example, we compute the homology of a polynomial's Milnor fiber using our result and a modification of the methods of [ST17], which are themselves based on Theorem 4.2.5:

Example 4.3.3 ([Hof])**.** Consider the holomorphic function $f : \mathbb{C}^3 \to \mathbb{C}$ given by the polynomial equation $f(x, y, z) = x^3 + xy^2z$ and the two-parameter deformation given by $F((x, y, z), s, t) = (x^2 + y^2z - s)(x - t)$. We will use this deformation to compute the Milnor fiber of f at the origin.

First note that, since F is defined over $\mathbb{Q}$, the Jacobian ideal $J_{F\times\pi} = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$ is as well. Using a computer algebra system such as Macaulay2, Singular, or Oscar, we find by direct computation that, for $R = \mathbb{Q}[x, y, z, s, t]$, the normal cone $C_{\operatorname{Spec} R/J_{F\times\pi}} \operatorname{Spec} R$ is given by $\operatorname{Spec}(R/J_{F\times\pi})[w_0, w_1, w_2]/(yw_1 - zw_2)$, and this is flat over the parameter space $\operatorname{Spec} \mathbb{Q}[s, t]$ at the origin. By the faithful flatness of field extensions, the construction of the normal cone and flatness result are preserved under base change to $\mathbb{C}$ (for the normal cone, see Lemma 2.3.15). We can then use Proposition 2.1.6 to show that they are preserved under analytification as well.

Hence we can conclude by Theorem 4.3.1 that this deformation is tame — by Theorem 4.2.4, it then defines a local fibration over the complement of the discriminant closure $\bar{\Delta}$. By computing the kernel of the map $\mathbb{Q}[v, s, t] \to R/J_{F\times\pi}$ with $v \mapsto F$ and again using our flatness results, we can see that $\bar{\Delta}$ is defined in $\mathbb{C} \times \mathbb{C}^2$ by the equation $4vst^4 - 8vs^2t^2 - 4v^2t^3 + 4vs^3 + 36v^2st - 27v^3$, where v is the coordinate on the target $\mathbb{C}$ of F. It now suffices, for $\varepsilon > 0$ any Milnor radius for f at the origin, to compute the intersection with $B_\varepsilon \times \mathbb{C}^2$ of the fiber of $F \times \pi$ over any point outside this locus which is close enough to the origin with small enough parameter values s and t.

For the sake of simplicity, we restrict over the curve $s = 5t^2$ in the parameter space; let f_t denote the restriction of F over a given value of t. The closure of the discriminant is now given by $320vt^6 + 176v^2t^3 - 27v^3$; by computing the irreducible components of the curve thus defined, we can now see that, for small values of t, our critical values of f_t near the origin will be 0, $8t^3$, and $-\frac{40}{27}t^3$. We can now compute the individual contributions of each point to the reduced homology of the Milnor fiber by adapting the method of [ST17].

Specifically, we proceed as follows. Observe that, if we take the inverse image under f of a small enough disk D_δ around the origin and intersect it with B_ε, the result will be homotopy-equivalent to the fiber $f^{-1}(0) \cap B_\varepsilon$, hence contractible. By including the boundary and applying Lemma 3.1.14, we find that, for small enough $t \neq 0$, the space $\mathbb{E} := {f_t}^{-1}(D_\delta) \cap B_\varepsilon$ is contractible as well. Letting $\mathbb{F}$ be a smooth fiber of f_t in B_ε, we then find that $H_{*+1}(\mathbb{E}, \mathbb{F}) \cong \tilde{H}_*(\mathbb{F})$ by the long exact sequence in homology, and as noted Theorem 4.3.1 and Theorem 4.2.4 tell us that $\tilde{H}_*(\mathbb{F})$ is precisely the reduced homology of the Milnor fiber of f.

Let D_0, D_+, D_- be sufficiently small disjoint open disks inside D_δ around the critical values 0, $8t^3$, and $-\frac{40}{27}t^3$ respectively. Observing that the transversal Milnor fibration given by fixing z is a local product at all points of the critical locus where $z \neq 0$, we find that the special points (in the sense of [ST17]) are exactly $q_+ := (-t, 0, 0)$ and $q_- := (\frac{5}{3}t, 0, 0)$; let B_+ and B_- be sufficiently small Milnor balls in B_ε around these points respectively. Also let $\mathcal{U}_0, \mathcal{U}_+, \mathcal{U}_-$ be sufficiently small tubular neighborhoods in B_ε of the connected components $\Sigma_0 := V(x - t, y^2 z - 4x^2) \cap B_\varepsilon, \Sigma_+ := V(x + t, y) \cap B_\varepsilon, \Sigma_- := V(3x - 5t, y) \cap B_\varepsilon$ of the critical locus, which we can compute by taking the minimal primes of $J_{F \times \pi}$. Indeed, by the argument in [Hof], we can compute the associated primes of J_{f_t} from those of $J_{F \times \pi}$ — see Figure 4.3 for a depiction of the critical locus in this case.

Then, if we let $\mathbb{E}_0 := {f_t}^{-1}(D_0) \cap \mathcal{U}_0$, $\mathbb{E}_+ := {f_t}^{-1}(D_+) \cap (\mathcal{U}_+ \cup B_+)$, and $\mathbb{E}_- := {f_t}^{-1}(D_-) \cap (\mathcal{U}_- \cup B_-)$ and $\mathbb{F}_0, \mathbb{F}_+, \mathbb{F}_-$ be the intersections of the smooth fibers of f_t over small non-critical values nearby the appropriate critical values with $\mathbb{E}_0, \mathbb{E}_+, \mathbb{E}_-$ respectively, we have

$$H_{*+1}(\mathbb{E}, \mathbb{F}) \cong H_{*+1}(\mathbb{E}_0, \mathbb{F}_0) \oplus H_{*+1}(\mathbb{E}_+, \mathbb{F}_+) \oplus H_{*+1}(\mathbb{E}_-, \mathbb{F}_-)$$

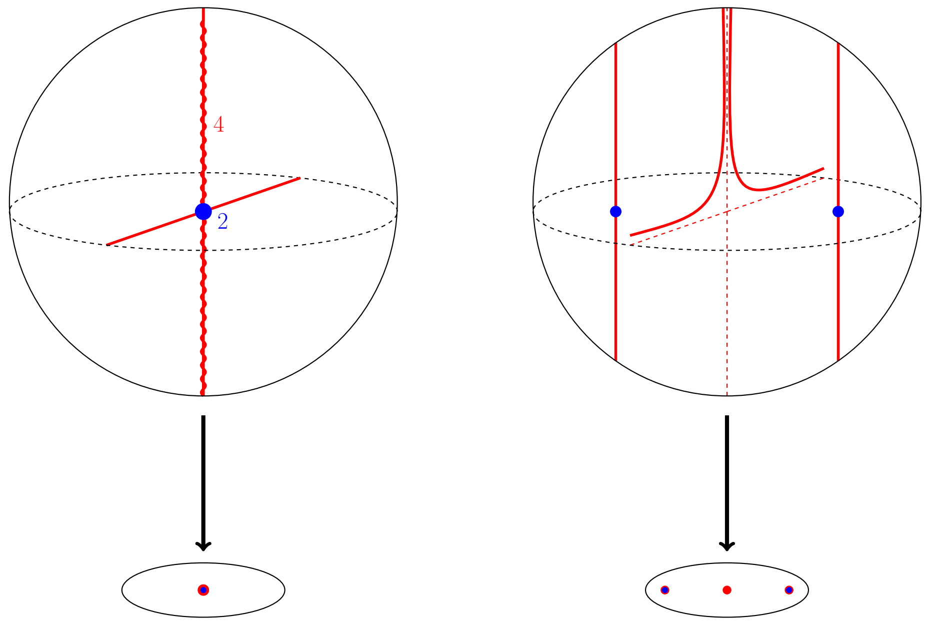


Figure 4.3: The critical loci of Example 4.3.3 over $t = 0$ and $t \neq 0$ respectively ($s = 5t^2$).

by homotopy retraction and excision.

We can now consider each of these relative homology groups independently. For each of the groups $H_{*+1}(\mathbb{E}_\pm, \mathbb{F}_\pm)$, we can take a further retraction of pairs to get the relative homology of ${f_t}^{-1}(D_\pm) \cap B_\pm$ and its intersection with a nearby smooth fiber; by our reasoning above, this is simply the reduced homology of the Milnor fiber of f_t at the point $q_\pm$. Since $x - t$ is a unit in the local power series ring at this point and multiplication by units does not affect the Milnor fiber, the point is a D_∞ singularity and hence the reduced homology of the Milnor fiber consists of a single $\mathbb{Z}$-summand in degree 2.

Consider the remaining component Σ_0 of the critical locus and note that $\mathbb{E}_0 \simeq \Sigma_0 \simeq S^1$. Thus, from the long exact sequence of a pair in homology and the fact that the bounds of [KM75] imply $H_{i+1}(\mathbb{E}, \mathbb{F}) \cong \tilde{H}_*(\mathbb{F}) \cong 0$ and hence $H_{i+1}(\mathbb{E}_0, \mathbb{F}_0) \cong 0$ for $i \notin \{1, 2\}$, we obtain a short exact sequence

$$0 \to H_2(\mathbb{E}_0, \mathbb{F}_0) \to H_1(\mathbb{F}_0) \to H_1(\mathbb{E}_0) \to 0$$

with $H_1(\mathbb{E}_0) \cong \mathbb{Z}$ and an isomorphism $H_3(\mathbb{E}_0, \mathbb{F}_0) \cong H_2(\mathbb{F}_0)$.

Now, by considering the transversal slices given by the level sets of the linear function y, we see that the transversal type of the singularity is everywhere A_1 on Σ_0 and the transversal Milnor fibration is a local product as before. Indeed, by noting $y \neq 0$ around Σ_0 and writing $f_t = (\tilde{x}^2 + \tilde{z} - 5t^2)(\tilde{x} - t)$ in the coordinates $(\tilde{x}, \tilde{y}, \tilde{z}) = (x, y, y^2 z)$, we obtain a global trivialization of the transversal Milnor fibration, so $\mathbb{F}_0$ is homotopy equivalent to a torus and the map $S^1 \times S^1 \simeq \mathbb{F}_0 \to \mathbb{E}_0 \simeq S^1$ is simply the projection onto one factor — here we are able to ignore any issues near the boundary of B_ε by an appropriate homotopy retraction.

As such, $H_3(\mathbb{E}_0, \mathbb{F}_0) \cong H_2(\mathbb{F}_0) \cong \mathbb{Z}$, and the map $\mathbb{Z} \oplus \mathbb{Z} \cong H_1(\mathbb{F}_0) \to H_1(\mathbb{E}_0) \cong \mathbb{Z}$ is the projection onto a single factor and hence has kernel $H_2(\mathbb{E}_0, \mathbb{F}_0) \cong \mathbb{Z}$. Combining this with our previous computation, we find that the reduced homology of the Milnor fiber of f is given by

$$\tilde{H}_i(\mathbb{F}) \cong \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Z}^{\oplus 3} & i = 2 \\ 0 & \text{else.} \end{cases}$$

This example demonstrates that Theorem 4.3.1 can be used to obtain computational results from deformations beyond the purview of the finite determinacy methods discussed in Subsection 4.2.2 and the Lê number-based approach of Subsection 4.2.3. In the case of finite determinacy, this is clear by our prior observation that unfoldings of the sort used in Theorem 2.2 of [Bob04] are holomorphically trivial locally away from the origin — in particular, their critical loci are as well, so this theorem is strictly less general than Theorem 4.3.1, which allows for any deformation of the embedding of the critical locus, not just the trivial one.

On the other hand, this deformation cannot have constant Lê numbers at the origin since the origin is no longer even a critical point of the deformed function for nonzero parameter values. As discussed in Section 8 of [Hof], the constancy of the Lê numbers is in general independent of the condition of Theorem 4.3.1.

4.3.2 Consequences for Homogeneous Polynomials

As discussed in Subsection 4.1.2, the Milnor fibration of a homogeneous polynomial captures, up to diffeomorphism, its global behavior over $\mathbb{C}^*$. In particular, since Proposition 4.1.10 implies that tame deformations preserve the Milnor fibration, Theorem 4.3.1 can be used to control the global behavior of a family of homogeneous polynomials:

Theorem 4.3.4 ([Hof]). *Let Y be a connected complex-analytic space, and*

$$F : \mathbb{C}^{n+1} \times Y \to \mathbb{C}$$

a holomorphic map such that, for each point $y \in Y$, the map $f_y := F|_{\mathbb{C}^{n+1}\times y}$ is a nonzero homogeneous degree-d polynomial. Then, if we let $\pi : \mathbb{C}^{n+1} \times Y \to Y$ be the projection, $\Sigma_{F\times\pi}$ is a closed subcone of $\mathbb{C}^{n+1} \times Y$ over Y; suppose that the natural projection $C_{\mathbb{P}\Sigma_{F\times\pi}}(\mathbb{P}^n \times Y) \to Y$ of the normal cone to its projectivization in the ambient trivial projective bundle is flat. Then the fibrations over $\mathbb{C}^*$ induced by the f_y as y varies throughout Y are all diffeomorphically equivalent.

Proof We first observe that, by taking the reduction Y_{red} of Y and pulling back along a resolution of singularities (see [Wło09]) $\tilde{Y}_{\text{red}} \to Y_{\text{red}}$, we may assume Y is smooth — this may result in a parameter space $\tilde{Y}_{\text{red}}$ with multiple connected components, but, since the fibration over any point of $\tilde{Y}$ is the same as the one over the corresponding point of the connected space Y, proving the constancy along each connected component will be sufficient to get consistency everywhere. Since the formation of the ideal $\mathcal{J}_{F\times\pi}$ commutes with pullback along maps into Y in general and the flatness hypothesis guarantees that the formation of the normal cone will do so as well by Lemma 2.3.15, the passage from Y to $\tilde{Y}_{\text{red}}$ does not affect our hypotheses.

Now observe that the complement $\Sigma_{F\times\pi} \setminus (0 \times Y)$ of the zero section in the critical locus is pullback of $\mathbb{P}\Sigma_{F\times\pi}$ along the natural $\mathbb{C}^*$-bundle map $(\mathbb{C}^{n+1} \times Y) \setminus$

$(0 \times Y) \to \mathbb{P}^n \times Y$ given by the quotient. In particular, since the bundle map is flat, the formation of the normal cone again commutes with this pullback by Lemma 2.3.15, and hence the flatness of $C_{\mathbb{P}\Sigma_{F\times\pi}}(\mathbb{P}^n \times Y)$ over Y implies that of $C_{\Sigma_{F\times\pi}\setminus(Y\times 0)}((\mathbb{C}^{n+1} \times Y) \setminus (0 \times Y))$. Since flatness is a local property and the formation of the normal cone commutes with localization, this is exactly to say that $C_{\Sigma_{F\times\pi}}(\mathbb{C}^{n+1} \times Y)$ is flat over Y everywhere on $\mathbb{C}^{n+1} \times Y$ except possibly at $0 \times Y$.

Then, since the discriminant $(F \times \pi)(\Sigma_{F\times\pi})$ in this case will simply be $0 \times Y \subset \mathbb{C} \times Y$ by Proposition 4.1.7, we can apply Theorem 4.3.1 and Theorem 4.2.4 to obtain, for each point $y \in Y$, a small neighborhood N_y of y and small enough values $\varepsilon_y, \delta_y > 0$ such that the maps

$$f_{y'} : B_{\varepsilon_y} \cap {f_{y'}}^{-1}(D^*_{\delta_y}) \to D^*_{\delta_y}$$

as y' varies in N_y fit together into a smooth locally trivial fibration over $D^*_{\delta_y} \times N_y$. In particular, these maps define smooth locally trivial fibrations over the punctured disk which are all diffeomorphically equivalent.

However, since any positive number is a Milnor radius for a homogeneous polynomial at the origin by Proposition 4.1.10, the fibrations we have defined for the $f_{y'}$ in the ball B_{ε_y} are, in fact, their Milnor fibrations at the origin, and hence diffeomorphic to their affine Milnor fibrations by Proposition 4.1.9. As such, the diffeomorphism type of the affine Milnor fibration of f_y is locally constant on Y, hence constant over all of Y if we assume connectedness. As discussed, the result follows.

We can apply this result to obtain a clearer picture of how the Milnor fibration varies over the space of all homogeneous polynomials:

Corollary 4.3.5. *Let $H_{n,d} \cong \mathbb{P}^{\binom{n+d}{n}-1}$ be the space of degree-d hypersurfaces in $\mathbb{P}^n$, so that the $\binom{n+d}{n}$ projective coordinates of each point give the coefficients up to scaling of the monomial terms in a homogeneous polynomial defining the corresponding hypersurface. Then iteratively applying Theorem 4.3.4 gives us a partition of $H_{n,d}$ into finitely many disjoint Zariski-locally-closed subsets so that the fiber diffeomorphism type of the affine*

Milnor fibrations of the corresponding defining polynomials is constant along each subset.

The proof of this result, which can be found in [Hof], is by viewing $H_{n,d}$ as the projectivization of the space of all degree-d polynomials and applying Theorem 4.3.4 repeatedly to the universal family of degree-d polynomials over this space and its restriction to successively-chosen closed subspaces thereof, defined by failures of flatness of the appropriate normal cone.

4.3.3 Consequences for Functions with Critical Locus a Complete Intersection

Consider the following expansion of Definition 2.4.5:

Definition 4.3.6. Let M be a smooth complex-analytic space and X a closed subspace of M, with $\mathcal{I}$ the corresponding ideal sheaf. Let $c := \dim M - \dim X$ and suppose that $\mathcal{I}$ is generated by c globally-defined holomorphic functions on M. Then we say that X is a **complete intersection** of codimension c in M.

By using the map $M \to \mathbb{C}^c$ defined by such a choice of generators, applying Theorem 2.4.4 at each point of X, and passing to the appropriate completion, we can see that the local ring of such a complex-analytic space at any point will be a local complete intersection ring in the sense of Definition 2.4.5. Note that the flatness of the map $M \to \mathbb{C}^c$ implies also that $X \hookrightarrow M$ is a regular embedding by Theorem 2.4.1.

One of the main restrictions which appears frequently in the literature surrounding the finite determinacy methods of Subsection 4.2.2 is the hypothesis that the function to be deformed have a critical locus which is a complete intersection in the ambient space at least set-theoretically (e.g., [Sie83; Sie87; Sie88; Pel85; Pel88; Pel89; Pel90; Zah94; Ném99; Gaf07]). If we require this to be true on the level of complex-analytic spaces, at least away from the origin, we find that Theorem 4.3.1 can be used to show that any deformation of such a function which respects the complete intersection structure is tame:

Proposition 4.3.7 ([Hof]; cf. [Zah94; Gaf07])**.** *Let $I = (g_1, \ldots, g_c) \subset \mathcal{O}_{\mathbb{C}^{n+1},0}$ be an ideal defining a germ at the origin of a complete intersection of codimension c in $\mathbb{C}^{n+1}$. Let*

$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic function germ such that the Jacobian ideal $J_f := (\mathcal{J}_f)_0$ is contained in I and $j(f) := \dim_{\mathbb{C}} I/J_f$ is finite.

Suppose we have germs $G_1, \ldots, G_c, F : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}, 0)$ of deformations of $g_1, \ldots, g_c, f$ respectively over a fixed parameter space $(\mathbb{C}^u, 0)$ such $J_{F\times\pi} := (\mathcal{J}_{F\times\pi})_0$ is contained in $\tilde{I} := (G_1, \ldots, G_c)$, where $\pi : (\mathbb{C}^{n+1+u}, 0) \to (\mathbb{C}^u, 0)$ is the projection. Then the intersection with $\pi^{-1}(0)$ of the germ at the origin of the non-flat locus of the embedding $\Sigma_{F\times\pi} \hookrightarrow \mathbb{C}^{n+1+u}$ over $\mathbb{C}^u$ is set-theoretically contained in the origin.

In particular, (F, π) is tame.

Proof The last statement follows from the rest of the proposition by Theorem 4.3.1.

Let $(S, 0)$ be the complex-analytic space germ cut out in $(\mathbb{C}^{n+1+u}, 0)$ by $\tilde{I}$ — more or less by definition, this is a germ of a complete intersection of codimension c in the ambient space. Its flatness over $\mathbb{C}^u$ is immediate by Proposition 2.4.6; to relate this to our critical locus, consider the short exact sequence

$$0 \to \tilde{I}/J_{F\times\pi} \to \mathcal{O}_{\Sigma_{F\times\pi},0} \to \mathcal{O}_{S,0} \to 0$$

of $\mathcal{O}_{\mathbb{C}^{n+1+u},0}$-modules and observe by the long exact sequence in Tor and the flatness of $\mathcal{O}_{S,0}$ over $\mathcal{O}_{\mathbb{C}^u,0}$ that it remains exact on restriction over the origin in $\mathbb{C}^u$.

As such, the fiber over $0 \in \mathbb{C}^u$ of $\tilde{I}/J_{F\times\pi}$ is precisely I/J_f. Our requirement that $j(f) < \infty$ tells us exactly that I/J_f has finite length — in particular, its support is zero-dimensional, hence concentrated at the origin in $\mathbb{C}^{n+1} \times 0$, and indeed it is empty if we have $j(f) = 0$. As such, Nakayama's Lemma (e.g., Corollary 4.8 of [Eis04]) tells us for any sufficiently small representative that the stalks of $\tilde{I}/J_{F\times\pi}$ will be zero at all points of $\mathbb{C}^{n+1} \times 0$, except possibly at the origin if $j(f) > 0$. Thus $\Sigma_{F\times\pi}$ agrees with S as complex-analytic spaces at such points and so it will be enough to show that the normal cone to $(S, 0)$ in $(\mathbb{C}^{n+1+u}, 0)$ is flat over $(\mathbb{C}^u, 0)$. However, since $S \hookrightarrow \mathbb{C}^{n+1+u}$ is a regular embedding, the normal cone is a vector bundle and hence the result follows from the flatness of S.

Bibliography

[AGV88] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of Differentiable Maps Volume II. Monodromy and Asymptotics of Integrals*. Modern Birkhäuser Classics. Boston, Massachusetts: Birkhäuser, 1988.

[AM86] Reynir Axelsson and Jón Magnússon. “Complex-Analytic Cones”. In: *Mathematische Annalen* 273 (1986), pp. 601–627.

[BMM94] Joël Briançon, Philippe Maisonobe, and Michel Merle. “Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom”. In: *Inventiones Mathematicae* 117 (1994), pp. 531–550.

[Bob04] Javier Fernández de Bobadilla. “Relative Morsification Theory”. In: *Topology* 37 (July 2004), pp. 925–982. DOI: 10.1016/j.top.2003.11.001.

[Dim92] Alexandru Dimca. *Singularities and Topology of Hypersurfaces*. Universitext. New York, New York: Springer-Verlag, 1992.

[Eis04] David Eisenbud. *Commutative Algebra. with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics 150. New York, New York: Springer, 2004.

[Fis76] Gerd Fischer. *Complex Analytic Geometry*. Lecture Notes in Mathematics 538. Heidelberg, Germany: Springer-Verlag, 1976.

[Fri67] Jacques Frisch. “Points de Platitude d’un Morphisme d’Espaces Analytiques Complexes”. In: *Inventiones Mathematicae* 4 (1967), pp. 118–138.

[Ful98] William Fulton. *Intersection Theory*. 2nd ed. Heidelberg, Germany: Springer-Verlag, 1998.

[Gab81] Ofer Gabber. “The Integrability of the Characteristic Variety”. In: *American Journal of Mathematics* 103.3 (1981), pp. 445–468.

[Gaf07] Terence Gaffney. “The Multiplicity of Pairs of Modules and Hypersurface Singularities”. In: *Real and Complex Singularities*. Ed. by Jean-Paul Brasselet and Maria Aparecida Soares Ruas. Basel, Switzerland: Birkhäuser Basel, 2007, pp. 143–168. DOI: 10.1007/978-3-7643-7776-2_11.

[GM18] Terence Gaffney and David B. Massey. “A Lemma for Relative Conormals Spaces”. In: *International Journal of Mathematics* 29.12 (2018).

[GM88] Mark Goresky and Robert MacPherson. *Stratified Morse Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge / A Series of Modern Surveys in Mathematics 14. Heidelberg, Germany: Springer-Verlag, 1988. DOI: 10.1007/978-3-642-71714-7.

[GR19] Terence Gaffney and Antoni Rangachev. "The A_f Condition and Relative Conormal Spaces for Functions with Non-Vanishing Derivative". In: *International Journal of Mathematics* 30.10 (2019).

[GR65] Robert C. Gunning and Hugo Rossi. *Analytic Functions of Several Complex Variables*. Eaglewood Cliffs, New Jersey: Prentice-Hall, 1965.

[Gro61a] Alexander Grothendieck. "Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : II. Étude globale élémentaire de quelques classes de morphismes". In: *Publications mathématiques de l'I.H.É.S.* 8.1 (1961), pp. 5–222.

[Gro61b] Alexander Grothendieck. "Techniques de construction en géométrie analytique. V. Fibrés vectoriels, fibrés projectifs, fibrés en drapeaux". In: *Séminaire Henri Cartan* 13.1 (1961), pp. 1–15.

[Ham71] Helmut Hamm. "Lokale Topologische Eigenschaften Komplexer Räume". In: *Mathematische Annalen* 191 (1971), pp. 235–252.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics 52. New York, New York: Springer, 1977.

[Hir73] Heisuke Hironaka. "Subanalytic Sets". In: *Number Theory, Algebraic Geometry, and Commutative Algebra, in Honor of Yasuo Akizuki*. Tokyo, Japan: Kinokuniya Book Store Co., Ltd., 1973, pp. 453–493.

[Hir77] Heisuke Hironaka. "Stratification and Flatness". In: *Real and Complex Singularities, Oslo 1976*. Ed. by Per Holm. Alphen aan den Rijn, The Netherlands: Sijthoff & Noordhoff International Publishers, 1977, pp. 199–265.

[HM96] Jean-Pierre Henry and Michel Merle. "Fronces et Doubles Plis". In: *Compositio Mathematica* 101.1 (1996), pp. 21–54.

[HMS84] Jean-Pierre Henry, Michel Merle, and Claude Sabbah. "Sur la Condition de Thom Stricte pour un Morphisme Analytique Complexe". In: *Annales scientifiques de l'École Normale Supérieure* 4^{e} *série* 17.2 (1984), pp. 227–268.

[Hof] Alex Hof. "Milnor Fiber Consistency via Flatness". arXiv:2212.12807.

[Hou61] Christian Houzel. "Géométrie Analytique Locale, II. Théorie des Morphismes Finis". In: *Séminaire Henri Cartan* 13.2 (1961), pp. 1–22.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*. Progress in Mathematics 236. Boston, Massachusetts: Birkhäuser, 2008.

[Huy05] Daniel Huybrechts. *Complex Geometry. An Introduction*. Universitext. Heidelberg, Germany: Springer-Verlag, 2005.

[JST] Cezar Joiţa, Matteo Stockinger, and Mihai Tibăr. "Tame Deformations of Highly Singular Function Germs". arXiv:2310.07340.

[KM75] Mitsuyoshi Kato and Yukio Matsumoto. "On the Connectivity of the Milnor Fiber of a Holomorphic Function at a Critical Point". In: *Manifolds—Tokyo, 1973: Proceedings of the International Conference on Manifolds and Related Topics in Topology, Tokyo, 1973*. Ed. by Akio Hattori. Tokyo, Japan: Mathematical Society of Japan, 1975.

[KS94] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*. Grundlehren der mathematischen Wissenschaften: A Series of Comprehensive Studies in Mathematics 292. Heidelberg, Germany: Springer-Verlag, 1994.

[Lê73] Dũng Tráng Lê. “Calcul du Nombre de Cycles Évanouissants d’une Hypersurface Complexe”. In: *Annales de l’Institut Fourier* 23.4 (1973), pp. 261–270.

[Lê77] Dũng Tráng Lê. “Some Remarks on Relative Monodromy”. In: *Real and Complex Singularities, Oslo 1976*. Ed. by Per Holm. Alphen aan den Rijn, The Netherlands: Sijthoff & Noordhoff International Publishers, 1977, pp. 397–403.

[LNS20] Dũng Tráng Lê, Juan José Nuño-Ballesteros, and José Seade. “The Topology of the Milnor Fibration”. In: *Handbook of Geometry and Topology of Singularities I*. Ed. by José Luis Cisneros Molina, Dũng Tráng Lê, and José Seade. Cham, Switzerland: Springer, 2020. Chap. 6, pp. 321–388.

[Loo84] E. J. N. Looijenga. *Isolated Singular Points on Complete Intersections*. London Mathematical Society Lecture Note Series 77. Cambridge, United Kingdom: Cambridge University Press, 1984.

[LT88] Dũng Tráng Lê and Bernard Teissier. “Limites d’Espaces Tangents en Géométrie Analytique”. In: *Comentarii Mathematici Helvetici* 63 (1988), pp. 540–578.

[Mas95] David B. Massey. *Lê Cycles and Hypersurface Singularities*. Lecture Notes in Mathematics 1615. Heidelberg, Germany: Springer-Verlag, 1995.

[Mat12] John Mather. “Notes on Topological Stability”. In: *Bulletin of the American Mathematical Society* 49.4 (2012), pp. 475–506.

[Mat70] John Mather. *Notes on Topological Stability*. Lecture notes; Harvard University. 1970.

[Mat89] Hideyuki Matsumura. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics 8. Cambridge, United Kingdom: Cambridge University Press, 1989.

[Max19] Laurenţiu G. Maxim. *Intersection Homology & Perverse Sheaves. with Applications to Singularities*. Graduate Texts in Mathematics 281. Cham, Switzerland: Springer, 2019.

[Max20] Laurenţiu G. Maxim. “Notes on vanishing cycles and applications”. In: *Journal of the Australian Mathematical Society* 109.3 (2020), pp. 371–415. DOI: 10.1017/S1446788720000403.

[Mil68] John Milnor. *Singular Points of Complex Hypersurfaces*. Princeton, New Jersey: Princeton University Press and the University of Tokyo Press, 1968.

[Nar66] Raghavan Narasimhan. *Introduction to the Theory of Analytic Spaces*. Lecture Notes in Mathematics 25. Heidelberg, Germany: Springer-Verlag, 1966.

[Ném99] András Némethi. “Hypersurface Singularities with 2-Dimensional Critical Locus”. In: *Journal of the London Mathematical Society* 59.2 (1999), pp. 922–938.

[Pel85] Ruud Pellikaan. “Hypersurface Singularities and Jacobi Modules”. PhD thesis. Rijksuniversiteit Utrecht, 1985.

[Pel88] Ruud Pellikaan. “Finite Determinacy of Functions with Non-Isolated Singularities”. In: *Proceedings of the London Mathematical Society* 57 (1988), pp. 357–382. DOI: 10.1112/plms/s3-57.2.357.

[Pel89] Ruud Pellikaan. “Series of Isolated Singularities”. In: *Singularities*. Ed. by Richard Randell. Contemporary Mathematics 90. Providence, Rhode Island: American Mathematical Society, 1989, pp. 241–259.

[Pel90] Ruud Pellikaan. “Deformations of Hypersurfaces with a One-Dimensional Singular Locus”. In: *Journal of Pure and Applied Algebra* 67 (1990), pp. 49–71. DOI: 10.1016/0022-4049(90)90163-C.

[Ree57] David Rees. “The Grade of an Ideal or Module”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 53 (1957), pp. 28–42.

[Sea19] José Seade. “On Milnor’s fibration theorem and its offspring after 50 years”. In: *Bull. Amer. Math. Soc.* 56.2 (2019), pp. 281–348. DOI: 10.1090/bull/1654.

[Ser56] Jean-Pierre Serre. “Géométrie Algébrique et Géométrie Analytique”. In: *Annales de l’Institut Fourier* 6 (1956), pp. 1–42. DOI: 10.5802/aif.59.

[Sie83] Dirk Siersma. “Isolated Line Singularities”. In: *Proceedings of Symposia in Pure Mathematics* 40.2 (1983), pp. 485–496. DOI: 10.1090/pspum/040.2/713274.

[Sie87] Dirk Siersma. “Singularities with Critical Locus a 1-Dimensional Complete Intersection and Transversal Type A_1”. In: *Topology and its Applications* 27 (1987), pp. 51–73. DOI: 10.1016/0166-8641(87)90057-5.

[Sie88] Dirk Siersma. “Hypersurfaces with Singular Locus a Plane Curve and Transversal Type A_1”. In: *Banach Center Publications* 20 (1988), pp. 397–410. DOI: 10.4064/-20-1-397-410.

[ST17] Dirk Siersma and Mihai Tibăr. “Milnor Fibre Homology via Deformation”. In: *Singularities and Computer Algebra*. Ed. by Wolfram Decker, Gerhard Pfister, and Mathias Schulze. Cham, Switzerland: Springer International Publishing, 2017, pp. 305–322. DOI: 10.1007/978-3-319-28829-1_14.

[Sta23] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.

[Tei77] Bernard Teissier. “The Hunting of Invariants in the Geometry of Discriminants”. In: *Real and Complex Singularities, Oslo 1976*. Ed. by Per Holm. Alphen aan den Rijn, The Netherlands: Sijthoff & Noordhoff International Publishers, 1977, pp. 565–677.

[Tei82] Bernard Teissier. “Varietes Polaires II: Multiplicites Polaires, Sections Planes, et Conditions de Whitney”. In: *Algebraic Geometry*. Ed. by José Manuel Aroca et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 1982, pp. 314–491. ISBN: 978-3-540-39367-2.

[Tro20] David Trotman. “Stratification Theory”. In: *Handbook of Geometry and Topology of Singularities I*. Ed. by José Luis Cisneros Molina, Dũng Tráng Lê, and José Seade. Cham, Switzerland: Springer, 2020. Chap. 4, pp. 243–273.

[Vak23] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. Version July 31, 2023. Lecture notes; Stanford University. 2023. URL: `https://math.stanford.edu/~vakil/216blog/`.

[Wei94] Charles Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics 38. Cambridge, United Kingdom: Cambridge University Press, 1994.

[Wło09] Jarosław Włodarczyk. “Resolution of Singularities of Analytic Spaces”. In: *Proceedings of the Gökova Geometry-Topology Conference 2008*. Ed. by Selman Akbulut, Turgut Önder, and Ronald J. Stern. International Press, 2009, pp. 31–63.

[Zah94] Alexandru Zaharia. “Topological Properties of Certain Singularities with Critical Locus a 2-Dimensional Complete Intersection”. In: *Topology and its Applications* 60 (1994), pp. 153–171. DOI: `10.1016/0166-8641(94)00009-3`.