Pomona College

Senior Thesis in Mathematics

# Chain Partitions of the Boolean Lattice

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Submitted to Pomona College in Partial Fulfillment
of the Degree of Bachelor of Arts

April 18, 2018

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# Introduction

The goal of this thesis is the examination of recent results by István Tomon on the possibility of obtaining chain partitions of the Boolean lattice with certain desirable properties. Specifically, we explore Tomon's refinements to the previously known asymptotic approximations of the Füredi partition [21] and his improvements on the bounds of Lonc and Elzobi [4] for the smallest Boolean lattice with a partition into chains of some specified size [22]. For the reader unfamiliar with such terminology, we provide an overview of the background material necessary to understand Tomon's results.

The questions Tomon's work tries to answer largely concern finite sets. Sets in general are among the most basic foundational objects in modern mathematics, making up the basis for the widely-used Zermelo-Fraenkel axiom system. Intuitively, they may be thought of as collections of distinct elements; for example, $\{1,2,3,4\}$ is a set with four members, as is $\{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}$.

In spite of the relative simplicity with which they may be described, these objects have many surprising properties; for example, the result that the set of positive integers has, in some sense, the same 'number' of elements as the seemingly much larger set of all integers, although familiar to most mathematicians, is generally counterintuitive when one first encounters it.

Indeed, the theory of infinite sets is famously riddled with startling twists and subversions of expectation, from Russell's paradox to the Banach-Tarski theorem. By comparison, the study of finite sets may seem, at first blush, to be incredibly simple, even to the point of triviality. However, this is far from accurate.

As an example of a property of these sets which is not yet well-understood, consider the four-element set $\{1,2,3,4\}$. It is fairly straightforward to list its subsets; these are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, and $\{1,2,3,4\}$ itself, where $\emptyset$ denotes the set with no elements. If we so choose, we can draw these in such

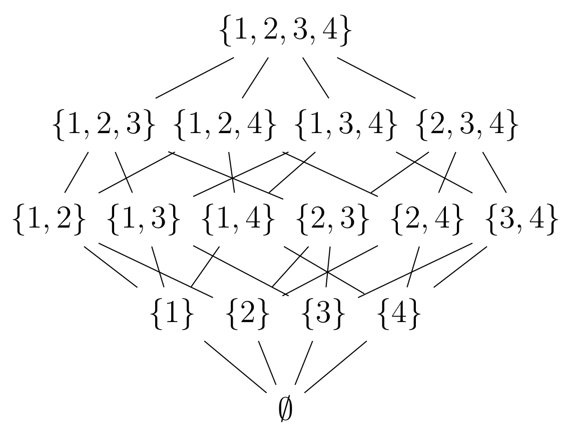


Figure 1: An illustration of the relationships between the subsets of $\{1, 2, 3, 4\}$.

a way that the subset relationships among them are more readily apparent, as depicted in Figure 1.

Suppose we want to write these subsets down in a way that preserves some of the information about their relations to one another without being as cumbersome as a diagram such as Figure 1. Then one option would be something like this: $\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\}$, $\{2\} \subset \{2, 3\} \subset \{2, 3, 4\}$, $\{3\} \subset \{3, 4\} \subset \{1, 3, 4\}$, $\{4\} \subset \{1, 4\} \subset \{1, 2, 4\}$, $\{1, 3\}$, and $\{2, 4\}$.

In listing these 'chains' of related subsets, we certainly do not convey everything which can be said about them; for example, very little about $\{1, 3\}$ and $\{2, 4\}$ is readily apparent. On the other hand, we get a lot more information at a glance than we did from our first list of subsets; the connections which this new list does express can be seen in Figure 2.

Now, it is clear that the way we have written out our chains is not the only way we could have done so; for example, we might have used $\emptyset \subset \{2\} \subset \{2, 3\} \subset \{2, 3, 4\} \subset \{1, 2, 3, 4\}$ and $\{1\} \subset \{1, 2\} \subset \{1, 2, 3\}$ instead of $\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\}$ and $\{2\} \subset \{2, 3\} \subset \{2, 3, 4\}$.

Less frivolously, were we dissatisfied with the lack of information about $\{1, 3\}$ and $\{2, 4\}$, we could write our subsets thus: $\emptyset \subset \{1\} \subset \{1, 3\}$, $\{1, 2\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\}$, $\{3\} \subset \{3, 4\} \subset \{1, 3, 4\}$, $\{4\} \subset \{1, 4\} \subset \{1, 2, 4\}$, $\{2\} \subset \{2, 3\}$, and $\{2, 4\} \subset \{2, 3, 4\}$. Figure 3 depicts the corresponding

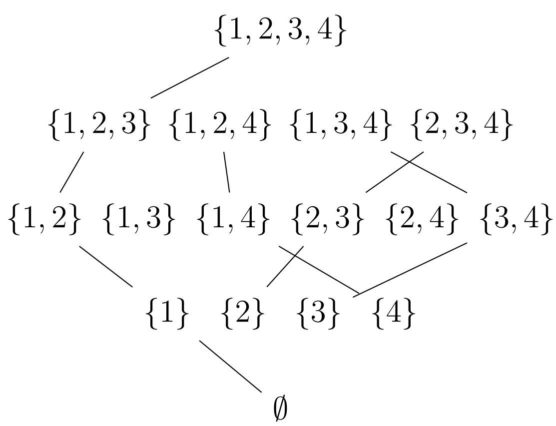

Figure 2: The connections apparent in our list of chains.

diagram.

The primary difference between these two lists of chains is the number of subsets included in each chain; the first has one chain with 5 subsets, three with 3, and two with 1, while the second has four with 3 and two with 2. This naturally raises the question of what sizes are possible. That is, given a list of positive integers summing to 16, the number of subsets of $\{1, 2, 3, 4\}$, can we group the subsets of the set into chains so that the given integers are the sizes of the chains?

Given the small size of $\{1, 2, 3, 4\}$, it is not terribly difficult to answer this question by brute force for any given list of positive integers. As we ask its equivalent for the subsets of larger sets, however, such computations become prohibitively expensive, and no general method of determining the answer without performing them is known. The goal of this thesis, as we have mentioned, is to examine some recent results of István Tomon in the study of this question.

The groundwork for many of the later explorations in this area was laid in the early- and mid-twentieth century by the mathematicians Emanuel Sperner [19] and Robert P. Dilworth [3], and the first result concerning our question in particular, which answered it in the affirmative for a specific list of sizes with a certain maximality property, was published independently by at least two authors, Kleitman [15] and Hansel [10], in the 1960s. The next step forward came in 1985 with the conjectures of Füredi [5] and Sands [18].

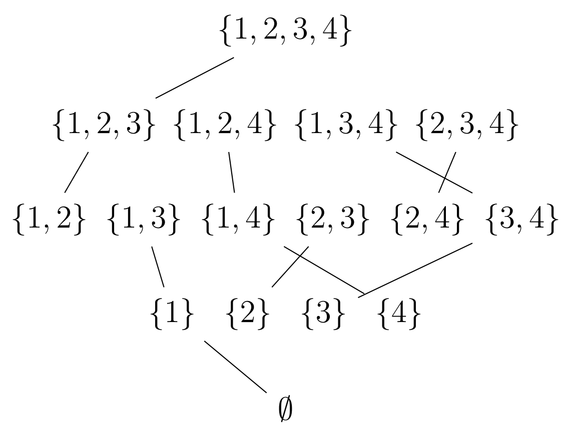


Figure 3: A different list of chains.

The Füredi conjecture, which remains open as of this writing, states that the subsets of a finite set can be broken down into a list of chains of the type we have been discussing such that the number of separate chains is minimal and the numbers of subsets in the chains differ pairwise by at most one. Recent progress on this conjecture has been made by Hsu, Logan, Shahriari, and Towse, who, in 2002, proved the possibility of obtaining a list with the minimum number of separate chains and some lower bound on the number of subsets in each chain [12] and, in 2003, expanded on this by developing a different method yielding both lower and upper bounds [13]. Hsu, Logan, and Shahriari also went on to prove an analogue of the conjecture for subspaces of certain finite-dimensional vector spaces in 2006 [14]. Most recently, in 2015, Tomon has created new methods which allow these bounds to be refined [21]; we shall examine these results later on.

The Sands conjecture, like the Füredi, concerns the possibility of writing chain lists with sizes satisfying some standard of uniformity. Specifically, it states that, for any positive integer power of 2, say $2^m$, there is a natural number so large that the subsets of any set of equal or greater size can be arranged in chains in such a way that each chain contains exactly $2^m$ subsets. This conjecture was generalized in 1988 by Pomona College alumnus Jerrold Griggs [9] to allow any positive integer $c$ in the place of $2^m$, with the natural stipulation that one of the chains be allowed to differ in size to account for the possibility that $c$ does not divide the number of subsets of any finite set. Shortly thereafter, in 1991, this generalized version of the conjecture was proven by Zbigniew Lonc [17], who went on to provide an explicit upper bound for the set size needed together with his student Muktar Elzobi in 2003 [4].

Although the generalized Sands conjecture has been proven, however, the upper bound on the size needed provided by Elzobi and Lonc appears to be a radical overestimate; for example, an earlier work of Griggs, Yeh, and Grinstead [8] established that the size needed for $c = 4$ is 9, many orders of magnitude smaller than $2^{2^{576}}$, the given upper bound. Therefore, obtaining estimates of the smallest size satisfying the conjecture remains an open area of inquiry. We shall examine some recent results of Tomon [22] which improve on the known upper bounds.

Three years after these conjectures arose, in 1988, Griggs proposed another [9], which generalizes that of Füredi and, if true, would dramatically simplify the task of improving on the bounds given by Elzobi and Lonc. This conjecture purports to give an exact characterization of whether it is possible to obtain a realization in chains of a given list of integers based upon an easily-checked property of that list. Although we will not cover the Griggs conjecture in detail, any reader with an interest in the topics we explore is encouraged to seek out and become acquainted with its formal statement.

The structure of the thesis is as follows. Chapter 1 introduces the basic concepts underlying the study of the Boolean lattice; the reader already familiar with the theory of normalized matching posets should be able to skip directly to Chapter 2, which formally introduces the Füredi conjecture and discusses Tomon's results [21] on the subject. Similarly, Chapter 3 gives the full statements of Sands' conjecture and Lonc's theorem before delving into Tomon's refined bounds [22]. Finally, Chapter 4 closes with a discussion of the results and some possible directions for future research.

# Chapter 1

# Preliminaries

Although we have already touched upon most of the basic concepts to be used in the results we will discuss, we have done so informally, with an eye to giving the reader a broad overview of the terrain to be covered. Here, we will pursue the ideas we need in greater depth, laying the groundwork for more rigorous discussion.

## 1.1 An introduction to posets

Previously, we considered the set of all subsets of $\{1, 2, 3, 4\}$, with a particular interest in which of them were subsets of others. This endowed the set of subsets with a kind of structure, which we illustrated in Figure 1. As it turns out, the precise properties of this type of structure can be formalized.

**Definition 1.1** Let $P$ be a set and $\leq$ a relation on the elements of $P$. Then we say that $\leq$ is a **partial order** if it satisfies the following three properties:

**Reflexivity:** $x \leq x$ for all $x \in P$

**Transitivity:** $x \leq y$ and $y \leq z$ together imply $x \leq z$ for all $x, y, z \in P$

**Antisymmetry:** $x \leq y$ and $y \leq x$ together imply $x = y$ for all $x, y \in P$

The pair $(P, \leq)$ is called a **partially ordered set**, or **poset** for short; in practice, the relation $\leq$ is often understood from context, and in these cases we refer to our poset simply as $P$.

Intuitively, we can understand these three properties by thinking of the usual ordering relation, uncoincidentally denoted $\leq$, on the real numbers. Through contemplation of this example, one can reasonably convince oneself that each of them is a fairly natural requirement for any relation which might plausibly be considered an ordering.

However, there is an important difference between a general poset and the set of real numbers. Specifically, our definition for a poset made no stipulation that every pair of elements be related in some way; whereas any pair of distinct real numbers has one element which is smaller than the other, in the case of a general poset it is entirely possible to have two distinct elements such that neither is less than the other. For example, in the inclusion relation $\subseteq$ on the subsets of $\{1, 2, 3, 4\}$ which we have been discussing, $\{1, 2\}$ and $\{3, 4\}$ are incomparable, since neither is a subset of the other. This possibility of incomparability is why such orders are called *partial*.

Although the interplay between partially ordered and so-called totally ordered sets such as the reals will be of interest to us later on, we focus for now on posets in and of themselves. In order to give the reader a concrete example to consider, and to make rigorous the notions which we have already discussed informally, we introduce the following paradigmatic poset.

**Example 1.2** Let $n$ be an element of the set $\mathbb{N}$ of positive integers. Then we denote by $[n]$ the finite set $\{1, 2, \ldots, n\}$. Let $2^{[n]}$ be the set of subsets of $[n]$ and $\subseteq$ the relation given by

$$A \subseteq B \Leftrightarrow B \text{ contains all elements of } A.$$

Then $(2^{[n]}, \subseteq)$ is a partially ordered set, which we refer to as the **Boolean lattice** on $n$ elements. As with other posets, we will usually write only $2^{[n]}$, leaving $\subseteq$ understood.

We have already examined $2^{[4]}$ in detail; $2^{[1]}$, $2^{[2]}$, and $2^{[3]}$ are depicted in Figures 1.1, 1.2, and 1.3 respectively. Note that, in the method of illustration we are using, called a **Hasse diagram** after the mathematician Helmut Hasse, we do not draw edges between all pairs of related vertices, instead relying on the reader's understanding that the relation is transitive to encapsulate the relationships between elements without visual clutter.

Keeping this family of examples in mind, we can now examine some basic properties of posets and additional definitions. We begin with a notion of what it means for two posets to be 'the same.'

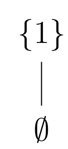


Figure 1.1: The Boolean lattice $2^{[1]}$.

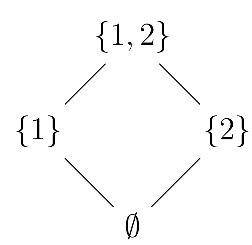


Figure 1.2: The Boolean lattice $2^{[2]}$.

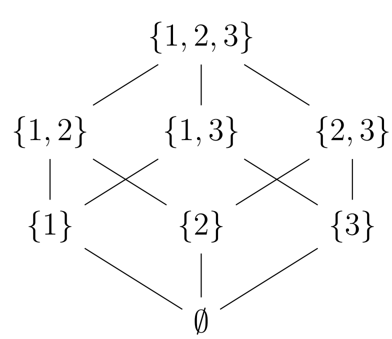


Figure 1.3: The Boolean lattice $2^{[3]}$.

**Definition 1.3** Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. Then we write $P \cong Q$ (or, more formally, $(P, \leq_P) \cong (Q, \leq_Q)$) if there exists an invertible function $\phi : P \to Q$ such that, for any $a, b \in P$,

$$a \leq_P b \;\Leftrightarrow\; \phi(a) \leq_Q \phi(b).$$

This corresponds to the intuitive notion that $P$ and $Q$ ought to be considered the same if they have the same structure up to some relabeling of their elements. In order to produce an example of this idea in action, we also introduce the following definition.

**Definition 1.4** Let $P$ be a poset with relation $\leq$. Then the **dual** $P^*$ of $P$ is the poset $(P, \leq_*)$, where, for all $a, b \in P$,

$$a \leq_* b \;\Leftrightarrow\; b \leq a.$$

That is, $P$'s dual has all of the same elements as $P$, and the relationships between them are precisely those obtained by reversing the relationships in $P$.

Note that this definition, strictly speaking, requires a proof of correctness. Specifically, we ought to show that the relation $\leq_*$ which we defined satisfies the definition of a partial order. However, since this proof is relatively straightforward, we elide it and proceed to the following result, which illustrates the concepts we have introduced thus far.

**Proposition 1.5** *Let $n$ be a positive integer. Then $2^{[n]} \cong (2^{[n]})^*$. That is, the Boolean lattices are self-dual.*

Intuitively, this fact can be seen in the vertical symmetry of the Boolean lattice diagrams we have drawn thus far, such as Figure 1.3. Although we will not pursue the proof in detail, we note that it begins by identifying each subset $A$ of $[n]$ with its complement $[n] \setminus A = \{k \in [n] \mid k \notin A\}$; interested readers are invited to work through the remaining details on their own.

Another fundamental notion is that of a subposet.

**Definition 1.6** Let $P$ be a poset with the relation $\leq$ and $Q$ a subset of $P$. Then $Q$ is a poset with the restriction of $\leq$ to elements of $Q$; we call this the **induced subposet of $P$ on $Q$**.

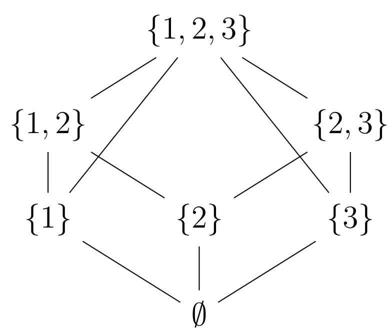


Figure 1.4: A subposet of $2^{[3]}$.

Although the terminology is somewhat long-winded, the basic concept is intuitive; if we wish to consider, for example, all subsets of [3] other than $\{1, 3\}$, it is still perfectly reasonable to think of the subset relationships between them, and this will indeed define a poset, which is illustrated in Figure 1.4.

Observe that, for any $n \in \mathbb{N}$, $[n] \subseteq [n+1]$ and so every subset of $[n]$ is also a subset of $[n+1]$. Therefore, we can view $2^{[n]}$ as a subposet of $2^{[n+1]}$ and, in fact, of $2^{[m]}$ for any $m \geq n$. However, more is true; this result can be strengthened using the notion of the product of two posets.

**Definition 1.7** Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. Then the Cartesian product $P \times Q$ is a poset, called simply the **product of $P$ and $Q$**, with the relation $\leq_{P\times Q}$ given by

$$(p_1, q_1) \leq_{P\times Q} (p_2, q_2) \;\Leftrightarrow\; p_1 \leq_P p_2 \text{ and } q_1 \leq_Q q_2.$$

We can now state precisely the relationship between $2^{[n]}$ and $2^{[n+1]}$.

**Proposition 1.8** *Let $n$ be a positive integer. Then $2^{[n+1]} \cong 2^{[n]} \times 2^{[1]}$.*

The essential concept of the proof of this fact is that we identify the elements of $2^{[n]} \times \{\emptyset\}$ with subsets of $[n+1]$ which do not contain $n+1$ and the elements of $2^{[n]} \times \{[1]\}$ with those which do; from here, it is a simple matter of verifying that the product relation is precisely the usual inclusion ordering. Figure 1.5 illustrates this correspondence for $n = 2$.

Thus $2^{[n]}$ is equivalent to the $n$th Cartesian power $(2^{[1]})^n$ of $2^{[1]}$ and, in fact, since $2^{[1]}$ is equivalent to [2] with the usual ordering on the integers, we

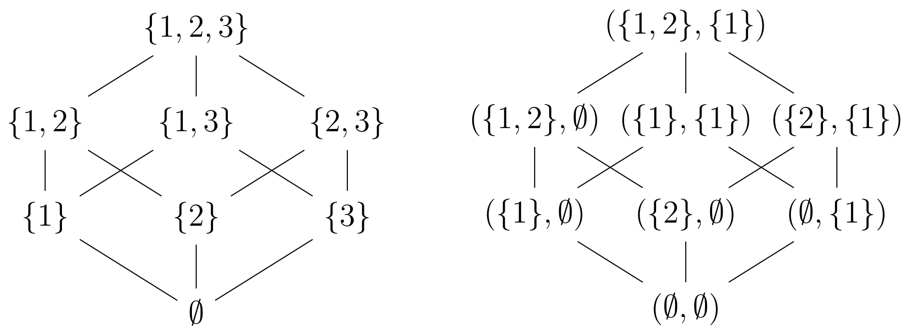


Figure 1.5: The Boolean lattice $2^{[3]}$, labeled both in the usual fashion and as the product $2^{[2]} \times 2^{[1]}$.

can see that $2^{[n]} \cong [2]^n$. This way of looking at the Boolean lattice situates it within a larger class of posets, which will be useful to us later on.

**Definition 1.9** Let $d$ be a positive integer. Then, if $k_1, k_2, \ldots, k_d$ are also positive integers, we say that the product poset $[k_1] \times [k_2] \times \ldots \times [k_d]$, where each $[k_i]$ has the usual ordering of its elements as integers, is a $d$-**dimensional grid**.

One possible 2-dimensional grid is depicted in Figure 1.6. Note that, under this definition, the Boolean lattice $2^{[n]} \cong [2]^n$ is simply one example of an $n$-dimensional grid; Figure 1.7 depicts $2^{[3]}$ as the grid $[2]^3$.

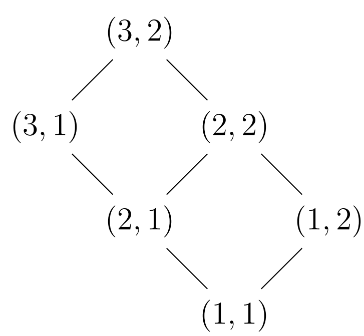


Figure 1.6: The 2-dimensional grid $[3] \times [2]$.

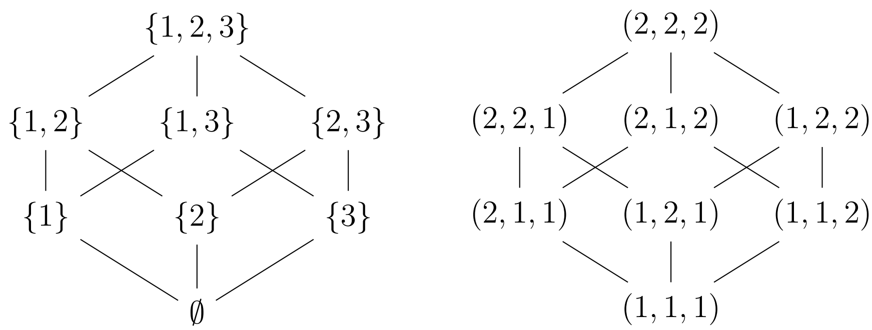


Figure 1.7: The Boolean lattice $2^{[3]}$ and the 3-dimensional grid $[2]^3$.

## 1.2 Chains and antichains

Posets, even those, such as the Boolean lattice, which have a nice characterization in terms of simpler objects, can have quite complicated structures which are difficult to reason about effectively. Therefore, it is often to our advantage to focus on simple substructures which will make it easier to reason about the poset as a whole.

### 1.2.1 Chains and ranked posets

The first of these which we will examine hearkens back to our earlier discussions of what makes a partial order 'partial' — namely, the possibility of pairwise incomparable elements. At that time, we noted that the set of real numbers is not a prototypical example of a poset precisely because it lacks such pairs of elements. As it turns out, posets with this property have two names, both of which may be somewhat unsurprising to the reader.

**Definition 1.10** Let $P$ be a poset. Then, if, for every $a, b \in P$, at least one of $a \leq b$ and $b \leq a$ is true, we say that $P$ is **totally ordered**, or a **chain**. The latter term is used most often, although not exclusively, when we are considering $P$ as a subposet of some larger poset $Q$.

Using this definition, we can now express our earlier concept of a grid more simply; specifically, we can define a $d$-dimensional grid as any product of $d$ finite chains. In addition, we can formalize our previous discussion of writing $2^{[4]}$ as a 'list of chains'.

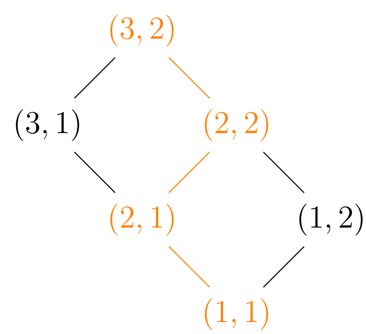


Figure 1.8: A chain in $[3] \times [2]$, highlighted in orange.

**Definition 1.11** Let $P$ be a poset. Then a **chain partition** of $P$ is a collection of subsets of $P$ such that:

- Each subset is a chain.
- The subsets are pairwise disjoint.
- The union of all the subsets is $P$ itself.

As we have noted, the primary goal of our work will be to examine the possibility of obtaining chain partitions of the Boolean lattice with certain specified chain sizes.

However, this is far from the only reason we have to concern ourselves with chains; these objects have a range of useful applications which allow us to better understand the structure inherent in the posets we will be working with. To examine one of the most powerful of these, we will need a notion of maximality for chains which are contained in larger posets.

**Definition 1.12** Let $P$ be a poset, and $M \subseteq P$ a chain. Then we say that $M$ is **maximal** if there is no element of $P \setminus M$ which is related to every element of $M$; that is, it is not possible to include any additional element in $M$ without losing its total ordering.

For example, the chain depicted in Figure 1.8 is maximal, since each of the elements outside it is incomparable to an element within it.

We can now examine the properties of a poset's maximal chains in the hope that this will allow us to get some idea of its internal structure. It turns out that, in many important instances, this is the case; that is, there are many interesting and widely-studied posets which satisfy the following property.

**Definition 1.13** Let $P$ be a finite, nonempty poset. Then we say that $P$ is **ranked** if all of its maximal chains contain the same number of elements. Since $P$ is nonempty, this number of elements will be equal to $\ell + 1$ for some nonnegative integer $\ell$; we say that $\ell$ is the **rank** of $P$.

In a ranked poset, the uniformity of the maximal chain sizes allows us to classify the poset's elements using their positions on the maximal chains containing them.

**Definition 1.14** Let $P$ be a ranked poset and consider $x \in P$. Then the **rank** of $x$, which we denote $\mathrm{rk}(x)$, is the number of elements strictly below $x$ on any maximal chain which contains it.

This notion of rank for an element, rather than a poset, is not, at its face, clearly well-defined. To convince ourselves that it is, we must observe that, were $x$ a member of two different maximal chains $C_1$ and $C_2$ such that the sets $B_1 = \{b \in C_1 \mid b < x\}$ and $B_2 = \{b \in C_2 \mid b < x\}$ contained differing numbers of elements, say $|B_1| > |B_2|$, we could construct a chain $B_1 \cup (C_2 \setminus B_2)$ with more elements than either of the maximal chains $C_1$ and $C_2$. Since it is not difficult to construct a maximal chain containing any given chain in a finite poset, this would contradict the uniformity of the sizes of $P$'s maximal chains, so our definition for rank is well-formed.

Having developed this classification for the elements of a ranked poset, it is natural to talk about the set of all elements sharing a given rank, which we call a level of the poset.

**Definition 1.15** Let $P$ be a poset of rank $n$. Then, for any integer $0 \leq i \leq n$, we say that the set $A_i = \{x \in P \mid \mathrm{rk}(x) = i\}$ is the ***i*th level of** $P$. We call the level sizes $|A_0|, |A_1|, \ldots, |A_n|$ the **rank numbers** of $P$.

As an example of these concepts at work, we return to the 2-dimensional grid $[3] \times [2]$ which we considered earlier in Figures 1.6 and 1.8.

**Example 1.16** Let $P = [3] \times [2]$. Then $P$ is a rank-3 poset with rank numbers 1, 2, 2, 1 and respective levels $A_0 = \{(1,1)\}$, $A_1 = \{(2,1),(1,2)\}$, $A_2 = \{(3,1),(2,2)\}$, and $A_3 = \{(3,2)\}$.

As one might expect, it is far from a coincidence that $[3] \times [2]$ is ranked. In fact, it is not terribly difficult to show that *any* grid has this property.

**Proposition 1.17** *Let $d$ be a positive integer and $P = [k_1] \times \ldots \times [k_d]$ a $d$-dimensional grid. Then $P$ is a ranked poset with rank function given by*

$$\mathrm{rk}(x_1, \ldots, x_d) = \sum_{i=1}^{d} (x_i - 1).$$

*In particular, since we can see from the definitions that the rank of a poset will be the maximum of the ranks of its elements, the rank of $P$ will be $\sum_{i=1}^{d} (k_i - 1)$.*

This is to say that the rank of an element $x = (x_1, \ldots, x_d)$ is the number of steps it takes, moving one unit in one of the coordinate directions at a time, to reach $x$ from $(1, 1, \ldots, 1)$, the lowest point in the grid. Since we are particularly interested in Boolean lattices, we specialize this result as follows.

**Corollary 1.18** *Let $n$ be a positive integer. Then $2^{[n]}$ is a ranked poset such that, for every $A \subseteq [n]$, $\mathrm{rk}(A) = |A|$, the number of elements in $A$. In particular, the rank of the entire poset is $n$ and, for every integer $0 \leq i \leq n$, the size of the poset's $i$th level is $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.*

This follows from our identification of $2^{[n]}$ with $[2]^n$; note that the formula for the rank is correct because our representation $x$ of a subset $A \subseteq [n]$ is such that the $i$th coordinate of $x$ is 2 instead of 1 if and only if $i \in A$. The result concerning the rank numbers is immediate from this because the number of $i$-element subsets of a set with $n$ elements is $\binom{n}{i}$.

Understanding a poset's rank numbers allows us to get a sense for its general shape. It is sometimes useful, for example, to consider posets which, without necessarily being self-dual, satisfy a less stringent symmetry constraint.

**Definition 1.19** Let $P$ be a ranked poset with levels $A_0, A_1, \ldots, A_n$. Then we say that $P$ is **rank-symmetric** if, for every integer $0 \leq i \leq n$, $|A_i| = |A_{n-i}|$.

This is to say that the bottom level contains the same number of elements as the top level, the first level above the bottom the same number as the first below the top, and so forth. Note that, since the Boolean lattices *are* self-dual, they necessarily satisfy the rank-symmetry condition as well. An argument similar to the one used to prove the result in the case of the Boolean lattices also demonstrates that any grid is self-dual and hence rank-symmetric.

Another important property which the shape of a ranked poset may possess is unimodality.

**Definition 1.20** Let $a_0, a_1, \dots, a_n$ be a finite sequence of real numbers. Then we say that this sequence is **unimodal** if there exists some integer index $0 \le r \le n$ such that

$$a_0 \le a_1 \le \dots \le a_{r-1} \le a_r \ge a_{r+1} \ge a_{r+2} \ge \dots \ge a_n;$$

that is, the sequence is strictly non-decreasing up to index $r$, after which it is strictly non-increasing instead.

Let $P$ be a ranked poset with levels $A_0, A_1, \dots, A_n$. Then we say that $P$ is **unimodal** if its sequence $|A_0|, |A_1|, \dots, |A_n|$ of rank numbers is.

Intuitively, we generally think of a unimodal poset as one which 'bulges' in the middle and tapers toward the top and bottom. It should be noted, however, that there are some cases not captured by this intuition; for example, it might be that the largest level $A_r$ is at the top or bottom, or that all the inequalities in our definition are actually equalities and hence every level of our poset is the same size. In such cases, the mental picture we have painted will not be entirely accurate, but it is still generally useful to keep in mind.

Every grid, including the Boolean lattices, which we have examined thus far has been unimodal. The following result generalizes this.

**Proposition 1.21** *Let $d$ be a positive integer and $P$ a $d$-dimensional grid. Then $P$ is unimodal.*

The reader may have an intuitive sense that the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ for any fixed $n$, and hence the Boolean lattices $2^{[n]}$, are unimodal. Indeed, it is not terribly difficult to prove this by induction on $n$ if one makes use of the recursive formula $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$ and the identity $\binom{n}{i} = \binom{n}{n-i}$, and the interested reader is invited to work through the details. In the case of a general grid, the recursive formula for the rank numbers is more involved and the argument correspondingly more difficult, although not impossible. In the interest of concision, we will not go into the details; for the unsatisfied, a proof using an alternate approach can be found in Anderson's *Combinatorics of Finite Sets* [1].

Note that the definition of a poset's unimodality is dependent on the index $r$ of some level of maximum size. Since we are interested in the Boolean lattices, we retrieve this parameter for such posets.

**Proposition 1.22** *Let $n$ be a positive integer. Then the only levels of maximum size in $2^{[n]}$ are the $\lfloor n/2 \rfloor$th and $\lceil n/2 \rceil$th.*

Here $\lfloor x \rfloor$ for a real number $x$ denotes the **floor** of $x$, which is defined to be the largest integer which is less than or equal to $x$. Similarly, $\lceil x \rceil$, the **ceiling** of $x$, is the least integer no smaller than $x$. Note that, in the case where $n$ is even, we will have $\lfloor n/2 \rfloor = n/2 = \lceil n/2 \rceil$, so there will be only one level of maximum size.

This result can be demonstrated by a modification of our earlier inductive argument, but the details are tiresome. A simpler approach is to use the formula $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ to consider the ratios of successive binomial coefficients. Indeed, this method actually provides an easier proof of the Boolean lattice's unimodality as well; however, it does not generalize to the case of an arbitrary grid as naturally as the inductive argument.

### 1.2.2 Antichains and width

Consider one of the levels, say the $i$th, of the Boolean lattice $2^{[n]}$. As we have seen, this level contains precisely those subsets of $[n]$ which have exactly $i$ elements. From this, it is apparent that no two distinct subsets on the level are comparable; were one a subset of another, it would have to be strictly smaller, violating the supposition that all have exactly $i$ elements. There is a name for a poset or subposet with this property.

**Definition 1.23** Let $P$ be a poset such that no two elements of $P$ are comparable to one another. Then $P$ is called an **antichain**.

As we have noted, any level of a Boolean lattice is an antichain. In fact, this result holds for a ranked poset in general.

**Proposition 1.24** *Let $P$ be a ranked poset with levels $A_0, A_1, \ldots, A_n$. Then each $A_i$ is an antichain in $P$.*

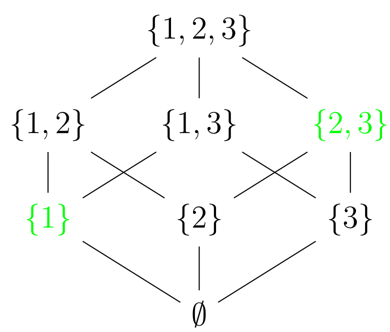


Figure 1.9: An antichain in $2^{[3]}$, highlighted in green.

**Proof** Suppose toward a contradiction that there exists $i \in \{0, 1, \ldots, n\}$ such that, for some $x, y \in A_i$, $x < y$. Since $P$ is finite, we can construct a maximal chain $M$ which contains all the elements in the chain $\{x, y\}$ by repeatedly adding in an element comparable to everything in our chain until no such element can be found. Since $x < y$ and the transitivity of $\leq$ implies that every element of $M$ strictly less than $x$ will also be strictly less than $y$, this tells us that the rank of $x$ is strictly less than that of $y$. However, this contradicts our hypothesis that $\mathrm{rk}(x) = \mathrm{rk}(y) = i$. ∎

However, the significance of antichains is not restricted to ranked posets alone. Indeed, one of the most fundamental properties of a poset is defined in terms of antichains.

**Definition 1.25** Let $P$ be a poset. Then the size of the largest antichain of $P$ is called the **width** of $P$.

Note that there is some question of whether the 'size of the largest antichain' is a well-formed object in the case of an infinite poset which contains arbitrarily large or infinite antichains, and in these cases greater specificity is needed. However, since our work is concerned only with finite posets, we deem such details beyond our scope and do not pursue them.

In the finite case, at least, we can see that this definition of width corresponds at least somewhat to our intuitions of what 'width' should be; for example, a chain, which we would expect from the name or definition to be long and skinny, will always have width exactly 1, whereas an antichain with $n$ members, which we would expect to be wider, will have width $n$.

The usefulness of width for our purposes, though, comes not from its correspondence with our intuitions per se but rather from its relation to chain partitions. We are interested in which collections of sizes for a chain partition are possible; as a first step, it is useful for us to know which *how many* chains such a partition might conceivably have. This question is the object of the following celebrated theorem of Dilworth.

**Theorem 1.26 ([3])** *Let $P$ be a poset of some finite width $w$. Then the minimum number of chains in any chain partition of $P$ is precisely $w$.*

It is clear that this minimum number of chains must be at least the width $w$, since there is an antichain with $w$ members; these elements are pairwise unrelated, so no two of them may be contained in a single chain and hence any chain partition of $P$ must have, at minimum, one chain for each of them. The content of Dilworth's theorem, which we will not attempt to prove here, lies in the fact that there is a partition of $P$ into exactly $w$ chains.

Although this gives us a nice starting point for thinking about the chain partitions which are and are not possible, attempting make use of it brings us to the difficulty of actually determining our posets' widths. In the case of the Boolean lattice $2^{[n]}$ in particular, it is not immediately clear how we might prove a general formula for the width in terms of $n$. As it happens, such a proof is possible, and was produced by Sperner in 1927 [19]; however, rather than attacking this problem directly, we will approach it in more general terms which also give us the opportunity to introduce an important family of posets.

## 1.3 Graphs and normalized matching posets

We now step back briefly from our examination of posets to introduce a related family of combinatorial objects, the graphs. There are actually several differing notions of what it means to be a graph; in the technical terminology, we will define our graphs so as to be of the sort which are finite, undirected, and simple, although we will refer to them as 'graphs' without reference to these adjectives. The reader unfamiliar with such distinctions may safely disregard them.

**Definition 1.27** Let $V$ be a finite set and $E$ a subset of the set of unordered pairs of elements of $V$. Then the ordered pair $G = (V, E)$ is called a **graph**.

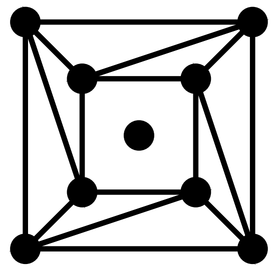

Figure 1.10: A graph with 9 vertices and 16 edges.

The elements of $V$ are called $G$'s **vertices** and, for $x, y \in V$, the unordered pair $\{x, y\}$, which is often written simply $xy$, is called an **edge** between $x$ and $y$ if it is in $E$. We may also say that an edge $xy$ is **incident** to $x$ and $y$ and, given that there is an edge between $x$ and $y$, we say that $x$ and $y$ are **adjacent** or that $x$ is a **neighbor** of $y$ (and vice versa).

Note that an unordered pair of elements in $V$ is really just a subset of $V$ containing exactly two elements. A graph is often visualized as a collection of points with line segments drawn between them to represent edges; one such visualization is depicted in Figure 1.10.

As in the case of a poset, a graph may possess a complicated internal structure which makes it difficult to reason about in its totality. Therefore, again as before, we can focus on simpler substructures which make it easier to comprehend at least some of the information contained in the graph as a whole. For example, we might, in a way similar to our previous approach of ignoring some relationships among elements of a poset to obtain a chain partition, choose to focus on some subset of the edges of a graph.

**Definition 1.28** Let $G = (V, E)$ be a graph. Then a **matching** of $G$ is a subset $M \subseteq E$ of the set of edges such that every vertex in $V$ is incident to at most 1 of the edges of $M$.

An example of a matching on the graph of Figure 1.10 is depicted in Figure 1.11. The reason we call such a subset a 'matching' is because each of its edges uniquely matches its incident vertices to one another; however, it should be noted that not every vertex is guaranteed to be incident to an edge of the matching, so there may be vertices which $M$ does not match to any other. We introduce terminology to be able to speak more readily about

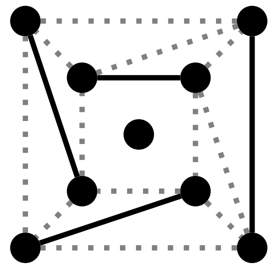

Figure 1.11: A matching on the graph of Figure 1.10. Unused edges are displayed in dotted gray.

such matters.

**Definition 1.29** Let $G = (V, E)$ be a graph, $M$ a matching of $G$, and $x \in V$. Then we say that $M$ **covers** $x$ if there is an edge incident to $x$ in $M$. Extending this notion, we also say that $M$ **covers** a set $X \subseteq V$ of vertices if it covers every element of $X$.

In using matchings to think about graphs, we will naturally be interested in those which cover as many vertices as possible, since these convey the most information about the graph. We formalize this notion as follows.

**Definition 1.30** Let $G = (V, E)$ be a graph and $M$ a matching of $G$. Then we say that $M$ is a **maximum matching** if there does not exist any matching $M'$ of $G$ which contains a strictly greater number of edges than $M$.

For example, the matching of Figure 1.11 is maximum, since the graph contains only 8 vertices which have incident edges and each edge in a matching must be incident to two unique vertices, meaning that no matching in such a graph may contain more than $8/2 = 4$ edges.

Note that, in spite of the similarity of the words in both form and informal meaning, it is not correct to use 'maximum' and 'maximal' interchangeably in this context. A matching is said to be **maximal** if no additional edges can be added to it to produce a new matching, and, although maximum matchings are clearly maximal, a maximal matching need not be maximum. This is to say that, with a little thought, we can construct a matching which cannot be extended by the inclusion of additional edges to a matching of maximum size. However, given a matching, it is not entirely impossible to construct a maximum one which corresponds to it in some meaningful way.

**Theorem 1.31** *Let $G = (V, E)$ be a graph and $M$ a matching of $G$. Then there exists some maximum matching $M'$ of $G$ which covers every vertex covered by $M$.*

Thus, although we cannot guarantee that there is a maximum matching containing the edges of a given matching $M$, we can at least produce one which does not leave any of the vertices matched in $M$ unmatched. This is a standard theorem of graph theory, although its proof is nontrivial and requires enough extraneous machinery that we will not touch upon it here. The curious reader may wish to seek it out in Asratian et al.'s *Bipartite Graphs and their Applications* [2].

Often, and especially in the study of matchings, we are interested in graphs where the edges encode a relationship between objects of two distinct types. For example, we might wonder whether, given a finite set of people and a finite set of fancy desserts such that each person would enjoy eating some, but not necessarily all, of the desserts, it is possible for us to give a dessert to each person such that all are satisfied. Formally, this problem amounts to searching for a matching $M$ of the graph with vertices which are either desserts or people, and an edge between two vertices exactly when one vertex is a dessert and the other is person who wants to eat that dessert, such that $M$ covers the set of vertices which correspond to people. The graphs which arise in such situations are called bipartite.

**Definition 1.32** Let $G = (V, E)$ be a graph. Then we say that $G$ is **bipartite** if there exist disjoint, nonempty subsets $X$ and $Y$ of $G$ such that $V$ is the union of $X$ and $Y$ and $E$ does not have any edge which connects two elements of $X$ or of $Y$ to one another.

When we wish to make explicit our choice of $X$ and $Y$ (since these are not guaranteed to be uniquely determined in all cases) and to emphasize that our graph is bipartite, we write $G = (X, \Delta, Y)$, where

$$\Delta = \{(x, y) \in X \times Y \mid xy \in E\},$$

instead of $G = (V, E)$.

An example bipartite graph is depicted in Figure 1.12. As we have mentioned, we are interested in finding matchings on bipartite graphs, particularly those which cover one of the two parts of the graph. The following theorem, first published in 1931 by König but often associated with Hall, who published it in 1935, gives a characterization for when this is possible.

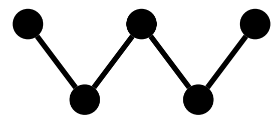

Figure 1.12: A bipartite graph, drawn so that elements in the same part have the same vertical position.

**Theorem 1.33 (Marriage Theorem)** *Let $G = (X, \Delta, Y)$ be a bipartite graph. Then $G$ has a matching which covers $X$ if and only if every $Z \subseteq X$ satisfies $|Z| \leq |\Gamma(Z)|$, where $\Gamma(Z) \subseteq Y$ denotes the set of neighbors in $G$ of elements of $Z$.*

The proof of this fact, and the origin of the theorem's name, are treated in Anderson's aforementioned work [1]. For our part, we are interested mostly in its implications for the bipartite graphs with the following important property.

**Definition 1.34** Let $G = (X, \Delta, Y)$ be a bipartite graph. Then $G$ is said to be **normalized matching** if, for every $Z \subseteq X$, we have

$$\frac{|\Gamma(Z)|}{|Y|} \geq \frac{|Z|}{|X|}.$$

That is, for any such $Z$, the proportion of $Z$'s neighbors within $Y$ is no smaller than the proportion of elements of $Z$ within $X$. For example, the graph of Figure 1.12 is normalized matching.

Note that our definition, as stated, appears to privilege one of the two parts $X$ and $Y$ of our graph over the other, in the sense that we do not require a similar condition for the neighbors in $X$ of a subset of $Y$. However, this is illusory; that is, $(X, \Delta, Y)$ is normalized matching if and only if $(Y, \Delta', X)$ for $\Delta' = \{(y, x) \mid (x, y) \in \Delta\}$ is. The proof of this fact is fairly straightforward.

We will, in our later work, find it to our advantage to be able to merge related normalized matching graphs to produce new ones. The following result guarantees that this is possible.

**Lemma 1.35** *Suppose $(X, \Delta_1, Y_1)$ and $(X, \Delta_2, Y_2)$ are normalized matching bipartite graphs with $Y_1$ and $Y_2$ disjoint. Then $(X, \Delta_1 \cup \Delta_2, Y_1 \cup Y_2)$ is normalized matching as well.*

That is, given two normalized matching bipartite graphs such that we can identify a part of one with a part of the other, we can combine the non-identified parts of the graphs with one another to produce a third graph, which also has the normalized matching property.

As in the case of bipartite graphs more generally, we may be interested in finding a matching on a bipartite graph with the normalized matching property which covers one of its parts. The following result on this topic is not difficult to prove using the Marriage Theorem.

**Proposition 1.36** *Let $G = (X, \Delta, Y)$ be a normalized matching bipartite graph. Then $G$ has a matching which covers $X$ if and only if $|X| \leq |Y|$.*

In particular, if both parts of our graph have the same size, the normalized matching property guarantees a matching which covers both, since the number of edges needed to cover the first will also necessarily cover the second. This fact will be of use to us later. For the time being, however, we would like to relate all of the concepts we have been building up back to our study of posets. The following construction allows us to do so.

**Definition 1.37** Let $P$ be a finite poset. Then the **comparability graph** $G(P)$ of $P$ is defined by $G = (P, E)$, where

$$E = \{\{x, y\} \mid x, y \in P \text{ and either } x < y \text{ or } y < x\}.$$

This is to say that $P$'s comparability graph is the graph which has vertex set $P$ and an edge between two distinct elements of $P$ if and only if the two are comparable.

We can apply this notion to the ranked posets with which we have been working. Specifically, we define a normalized matching property for posets as follows.

**Definition 1.38** Let $P$ be a ranked poset with levels $A_0, A_1, \ldots, A_n$. Then we say that $P$ is **normalized matching** if, for every choice of distinct $i, j \in \{0, \ldots, n\}$, the comparability graph $G(A_i \cup A_j)$ of the induced subposet on the levels $A_i$ and $A_j$ has the normalized matching property for bipartite graphs.

This is to say that, if we consider the levels of $P$ pairwise, the relationships between their elements will give us normalized matching graphs. Note that, by Proposition 1.24, each $A_i$ will be an antichain and so, for any choice of distinct $i$ and $j$, the induced subposet on the corresponding levels of $P$ will actually have a bipartite comparability graph of the form $(A_i, \Delta, A_j)$; hence, our definition is well-formed.

As the reader may be expecting, this notion is relevant to the classes of posets with which we are concerned. Specifically, the following result holds.

**Proposition 1.39** *Let $d$ be a positive integer and $P$ a $d$-dimensional grid. Then $P$ has the normalized matching property.*

In particular, the Boolean lattice $2^{[n]}$ is a normalized matching poset for any positive integer $n$. This fact has several proofs, requiring varying degrees of additional conceptual machinery and of effort; we again refer the interested reader to Anderson's work [1].

We, just before our digression into graph theory, evinced an interest in determining the width of the Boolean lattice; now, it will be possible to use the normalized matching property to do so. Consider the following property of a ranked poset.

**Definition 1.40** Let $P$ be a ranked poset with levels $A_0, A_1, \ldots, A_n$. Then we say that $P$ has the **LYM property** if, for every antichain $\mathcal{A} \subseteq P$,

$$\sum_{i=0}^{n} \frac{|\mathcal{A} \cap A_i|}{|A_i|} \leq 1.$$

This definition shares with that of the normalized matching property the idea that, rather than considering the *absolute* size of a subset of a ranked poset's level, we should really focus on its size as a proportion of the whole level. Therefore, it is reasonable to suspect that the two properties might be related. The following theorem of Kleitman characterizes the extent to which this is true.

**Theorem 1.41 ([16])** *Let $P$ be a ranked poset. Then $P$ is normalized matching if and only if it has the LYM property.*

From the LYM property, it is not terribly difficult to deduce the following fact.

**Corollary 1.42** *The width of a normalized matching poset is precisely the size of its largest level.*

Therefore, in the case of the Boolean lattices, we can use Proposition 1.22 to obtain an explicit formula for the width.

**Proposition 1.43** *Let $n$ be a positive integer. Then the width of $2^{[n]}$ is given by the binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$.*

We are now in a position to begin our examination of the outstanding conjectures on the Boolean lattice.

# Chapter 2

# Approximating the Füredi Partition

As we have mentioned, we are interested in Füredi's 1985 conjecture on the possibility of obtaining a particular partition of the Boolean lattice. We state it now in formal terms.

**Conjecture 2.1** *Let $n$ be a positive integer. Then $2^{[n]}$ has a partition into the minimum number of chains such that the chain sizes differ pairwise by at most 1.*

By Dilworth's theorem and our results on the width of the Boolean lattice, we know that this minimum number of chains is exactly $\binom{n}{\lfloor n/2 \rfloor}$. Hence, since $|2^{[n]}| = 2^n$, we can conclude that the chain sizes in such a partition will be $\ell$ and $\ell + 1$, where

$$\ell = \left\lfloor \frac{2^n}{\binom{n}{\lfloor n/2 \rfloor}} \right\rfloor,$$

and that the number of chains of size $\ell + 1$ will be the remainder of $2^n$ on division by $\binom{n}{\lfloor n/2 \rfloor}$. We call a partition of $2^{[n]}$ with these chain sizes a **Füredi partition**.

As one might infer from the fact that Füredi's conjecture has remained open for more than three decades, Füredi partitions are highly nontrivial to construct in general. Therefore, in the search for a method which might yield a proof of the conjecture, it is advantageous to settle for some loosening of these restrictions; that is, we construct chain partitions which, though they may have chain sizes differing by more than one, satisfy some weaker criterion

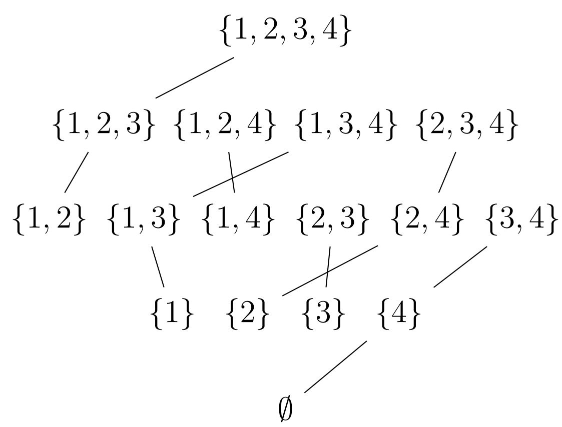


Figure 2.1: A Füredi partition of $2^{[4]}$. Note that the chain sizes are 3, 3, 3, 3, 2, and 2.

for approximate uniformity of chain sizes. We do so both for the sake of these partitions themselves, since it is entirely possible that they might be uniform enough for some applications, and in the hope that the methods we develop might be refined to yield a Füredi partition.

In this chapter, we will explicate the results of István Tomon's 2015 paper *On a conjecture of Füredi* [21], which introduces a variety of techniques for obtaining partitions into chains of approximately uniform size. Although we have occasionally made changes to the paper's reasoning to produce tighter or more precisely stated bounds and, in one case, introduced a new lemma (2.5) to address a gap in the original proof of Lemma 2.6, all subsequent results are due to Tomon.

## 2.1 Results on normalized matching posets

Note that, although we have followed Füredi in stating his conjecture in terms of the Boolean lattice, it is quite possible to ask whether a partition satisfying the same criteria exists for any given poset. That is, if $P$ is a poset of width $w$, we can ask whether $P$ has a partition into $w$ chains such that each has size $\left\lfloor \frac{|P|}{w} \right\rfloor$ or $\left\lfloor \frac{|P|}{w} \right\rfloor + 1$.

Therefore, various authors have proposed more general classes of posets for which the analogue of the Füredi conjecture might hold. Hsu et al. [14] suggest that it may be possible to obtain a Füredi partition for any rank-symmetric, unimodal normalized matching poset. Tomon, in the paper which we are examining, further generalizes this by positing that the rank-symmetry requirement is unnecessary, and the result holds for any unimodal normalized matching poset. Note that, since the Boolean lattice is rank-symmetric, unimodal, and normalized matching, either of these modified conjectures would imply the truth of the Füredi conjecture.

### 2.1.1 Partitions into chains with sizes bounded above

In pursuit of his favored generalization of Füredi's conjecture, Tomon introduces several results concerning unimodal normalized matching posets. The first of these states that we can partition such a poset into chains so that no chain is larger than around twice the size we would want for the Füredi partition.

**Theorem 2.2 ([21])** *Let $P$ be a unimodal normalized matching poset of width $w$. Then $P$ can be partitioned into $w$ chains of size strictly less than $\frac{2|P|}{w}+1$.*

Most of the work lies in the proof of the following lemma, which gives an analogous result for normalized matching posets which are **monotone** rather than just unimodal.

**Lemma 2.3 ([21])** *Let $P$ be a normalized matching poset of width $w$ with monotonically non-increasing rank numbers. Then $P$ can be partitioned into $w$ chains such that each is of size strictly less than $\frac{2|P|}{w}$.*

Note that this fact can be applied to show the equivalent result when the rank numbers are monotonically non-decreasing instead, since a chain partition of a poset is also a chain partition of its dual. The basic concept of the proof is that we can flatten some of the small levels higher up in the poset together and juxtapose this with several of the lower levels to obtain a new normalized matching poset with the same width and a smaller rank; using a chain partition of this new poset, we can construct one of the old poset with chain sizes bounded above.

**Proof** Observe that, if $|P| = w$, the result is trivial since partitioning $P$ into singletons gives us a partition into $w$ chains with chain sizes 1.

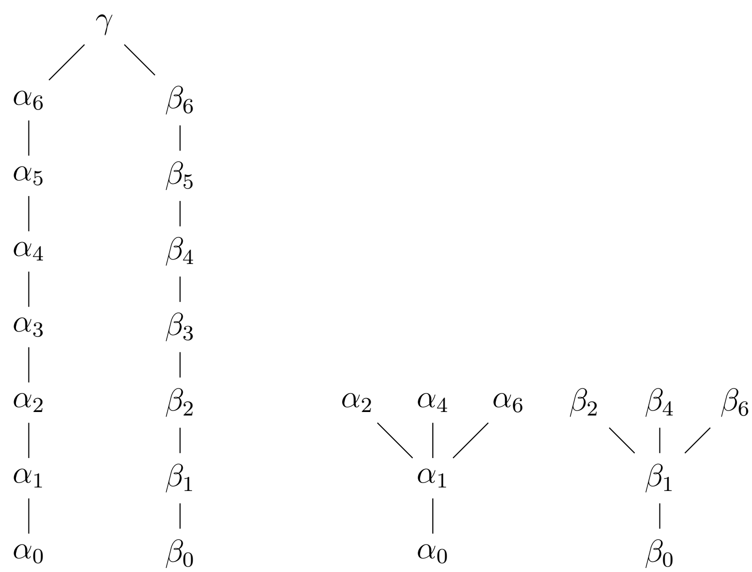


Figure 2.2: A rank-7 poset $P$ and its corresponding $Q_2$. Note that not all elements of $P$ are present in $Q_2$.

Otherwise, since $w$ is the size of $P$'s largest level by Corollary 1.42, $P$ has at least two levels, and so we can let $A_0, A_1, \ldots, A_n$ for some positive integer $n$ be $P$'s levels. By hypothesis, we have $w = |A_0| \geq |A_1| \geq \ldots \geq |A_n|$.

For $k \in \mathbb{N}$, let $Q_k$ be the rank-$k$ poset with levels $B_0 = A_0, B_1 = A_1, \ldots, B_{k-1} = A_{k-1}, B_k = \bigcup_{r=1}^{\lfloor n/k \rfloor} A_{rk}$, where the relationships between elements in different $B_i$ are induced by their relationships in $P$ and all the elements within each $B_i$ are pairwise unrelated. That is, $Q_k$ is the poset obtained by collapsing all levels whose indices are nonzero multiples of $k$ into the same level and removing every other level with index greater than $k$; this is depicted for a simple example poset in Figure 2.2. Lemma 1.35 on merging normalized matching graphs, applied repeatedly, tells us that $Q_k$ is normalized matching since $P$ is.

Now let $d$ be the smallest such $k$ satisfying $\operatorname{width}(Q_k) = w$, which is to say $\sum_{r=1}^{\lfloor n/k \rfloor} |A_{rk}| \leq w$, since the only way $Q_k$ can have a width other than $w$ is if the merged level is too large. Observe for any natural number $k$ and $r \in \{1, \ldots, \lfloor n/k \rfloor\}$ that the sum $|A_{rk-(k-1)}| + |A_{rk-(k-2)}| + \ldots + |A_{rk-1}| + |A_{rk}|$ of the sizes of $A_{rk}$ and the $k-1$ preceding levels of $P$ is, by $P$'s monotonicity, greater than or equal to $|A_{rk}| + |A_{rk}| + \ldots + |A_{rk}| + |A_{rk}| = k|A_{rk}|$. Thus $\sum_{r=1}^{\lfloor n/k \rfloor} |A_{rk}| \leq \frac{1}{k} \sum_{i=1}^{k\lfloor n/k \rfloor} |A_i|$, which is less than or equal to $\frac{1}{k}(|P| - |A_0|) = \frac{1}{k}(|P| - w)$ by the definitions. As such, a sufficient condition to obtain $\text{width}(Q_k) = w$ is $\frac{1}{k}(|P| - w) \leq w$, which is equivalent to $k+1 \geq \frac{|P|}{w}$, or $k \geq \frac{|P|}{w} - 1$. Since there is necessarily a positive integer in the range $\left[\frac{|P|}{w} - 1, \frac{|P|}{w}\right)$ by our stipulation that $|P| > w$, this gives us $d < \frac{|P|}{w}$.

Since $Q_d$ is of width $w$, Dilworth's theorem tells us it has a partition into $w$ chains, each of which clearly has size at most $d + 1$. Since all relations in $Q_d$ are valid in $P$, this gives us such a partition on the subposet of $P$ with the same elements as $Q_d$ but all of the induced relations, with the property that each element of any of $A_d, A_{2d}, \ldots, A_{d\lfloor n/d \rfloor}$ is the largest element of a distinct chain in the partition. Denote the set of chains of this partition by $\{C_i \mid 1 \leq i \leq w\}$.

For each $i \in \{1, \ldots, \lfloor n/d \rfloor\}$, define $R_i$ to be the induced subposet of $P$ on $\bigcup_{\rho=0}^{d-1} A_{id+\rho}$, where we define $A_j = \emptyset$ whenever $j > n$. This is to say that $R_i$ consists of the levels of $P$ with indices from $id$ to $(i+1)d - 1$, with the 'levels' which are above the top of $P$ defined to be empty. Then each $R_i$ is a normalized matching poset of $d$ or fewer levels with monotonically non-increasing sizes, so we can partition every $R_i$ into $|A_{id}|$ chains of size at most $d$, one for each element on the lowest level. Denote each such chain by $D_x$, where $x$ is the chain's minimal element.

Now we can, for each $1 \leq i \leq w$, define $C_i'$ such that, if $x \in C_i$ for some $x \in A_d \cup A_{2d} \cup \ldots \cup A_{d\lfloor n/d \rfloor}$, $C_i' = C_i \cup D_x$, and $C_i' = C_i$ otherwise. By our prior reasoning, such an $x$ will be the unique largest element of $C_i$ if it exists, so each $C_i'$ is, in fact, a chain. Since each such coupled $C_i$ and $D_x$ will have an intersection of exactly 1, we have $|C_i'| \leq (d+1) + d - 1 = 2d < \frac{2|P|}{w}$. Moreover, every element of $P$ will be in exactly one $C_i'$, since we have constructed these chains by stitching together chain partitions of subposets of $P$ in such a way that no chain is left out and no element is reused. Thus we can see that $\{C_i' \mid 1 \leq i \leq w\}$ is a partition of $P$ into $w$ chains of size strictly less than $\frac{2|P|}{w}$. ∎

With this result established, the proof of Theorem 2.2 is fairly straightforward. The basic concept is that we can view a unimodal poset as, essentially, two monotone posets stacked on top of one another and overlapping at one level. For example, in the case of the Boolean lattice $2^{[4]}$, these monotone posets are as depicted in Figure 2.3. The chain partitions of these halves

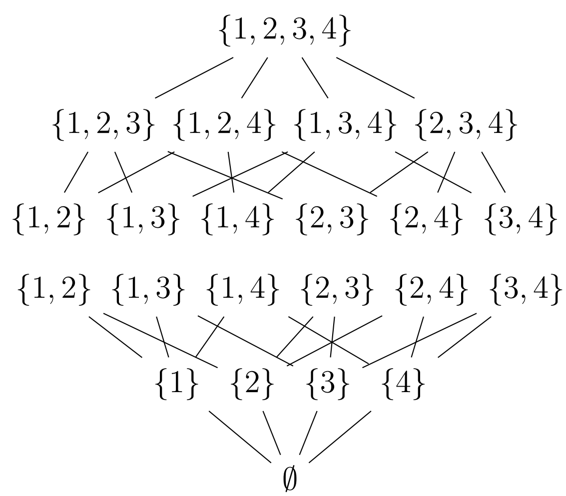


Figure 2.3: The two monotone halves of $2^{[4]}$. Note the shared level.

given by our lemma can then be combined to prove the theorem.

**Proof of Theorem 2.2** Consider a unimodal normalized matching poset $P$ of width $w$ with levels $A_0, A_1, \ldots, A_n$ and let $\ell \in \{0, \ldots n\}$ be such that $|A_\ell| = w$. If we let $U$ be the induced subposet of $P$ on all levels of index at least $\ell$ and $L$ the one on all levels of index at most $\ell$, we can observe that $|U| + |L| = |P| + w$, since the two subposets together contain all of $P$ and are disjoint except for a shared level of size $w$.

Since $U$ and $L$ are both posets of width $w$ with monotone level sizes, we can apply Lemma 2.3 to get chain partitions of each into $w$ chains with chain sizes strictly less than $\frac{2|U|}{w}$ and $\frac{2|L|}{w}$ respectively. Since each of the chains in each of these partitions will intersect $A_\ell$ at precisely one element, we can merge the chains $V_x \subseteq U$ and $M_x \subseteq L$ of each partition corresponding to the same element $x$ of $A_\ell$ to obtain a chain of size $|V_x| + |M_x| - |\{x\}| < \frac{2|U|}{w} + \frac{2|L|}{w} - 1 = \frac{2|P|+2w}{w} - 1 = \frac{2|P|}{w} + 1$. This gives us a chain partition of $P$ with the desired properties, so the theorem holds. ∎

Therefore, as promised, for any unimodal normalized matching poset, we can obtain a chain partition with sizes at most roughly a factor of two larger than those suggested by the generalized version of Füredi's conjecture.

### 2.1.2 Partitions into chains with sizes bounded below

We have already given a result which approaches the generalized Füredi conjecture by producing a chain partition with chains whose sizes cannot be too much larger than those of a Füredi partition. In the best case for this approach, if we were able to bring our bound down to guarantee a partition with chain sizes less than or equal to $\left\lfloor \frac{|P|}{w} \right\rfloor + 1$, we would not obtain a Füredi partition precisely, since the possibility would remain of having too many chains of this maximum size and hence having some chains smaller than $\left\lfloor \frac{|P|}{w} \right\rfloor$. However, this partition might still be useful in obtaining the Füredi, since we could reason about the possibility of reassociating elements of the excess long chains to obtain the sought-after partition.

On the other hand, even if we could produce such a bound, it is not *guaranteed* that a reassociation of the type we have discussed could be found. Thus, it might plausibly be more useful to instead bound our chain sizes from below. If we could increase such a bound to $\left\lfloor \frac{|P|}{w} \right\rfloor$, we would again not be guaranteed the Füredi partition, since we could have too many chains of the minimum size and hence some chains larger than $\left\lfloor \frac{|P|}{w} \right\rfloor + 1$, but the possibility that this reassociation problem would be more tractable than the other exists and, of course, it is far from clear which bound might be easier to push to the desired size.

Therefore, although it may seem redundant at first blush, obtaining a partition into chains with sizes which are not too much smaller than those of the Füredi partition is a reasonable avenue of inquiry to pursue. Tomon's result, again on unimodal normalized matching posets, gives a partition where the chain sizes are at least around half of the desired ones.

**Theorem 2.4 ([21])** *Let $P$ be a unimodal normalized matching poset of width $w > 1$. Then $P$ can be partitioned into $w$ chains of size at least $\frac{|P|-1}{2(w-1)} - \frac{1}{2}$.*

The proof of this fact relies on Lemmas 2.5 and 2.6, both of which are highly technical.

**Lemma 2.5** *Let $a_0 \geq a_1 \geq \ldots \geq a_m$ be positive integers such that, if we* *define* $f : \{0, \ldots, m\} \cup \{\infty\} \to \{0, \ldots, m\} \cup \{\infty\}$ *by*

$$f(k) = \begin{cases} \min\{k < i \leq m \mid a_{k+1} + \ldots + a_i \geq a_0\} & \textit{if this set is non-empty} \\ \infty & \textit{otherwise} \end{cases},$$

*there exists some* $p \in \{0, \ldots, m\}$ *with* $f(p) = m$. *Then, for any non-negative integers* $x_0, \ldots, x_p$ *such that* $x_i \leq a_i$ *for every* $i \in \{0, \ldots, p\}$, *we have*

$$\sum_{i=1}^{m} a_i \max\left\{ \frac{x_k}{a_k} \;\middle|\; k \in \{0, \ldots, p\}, k < i \leq f(k) \right\} \geq \sum_{i=0}^{p} x_i.$$

This result is somewhat abstruse, and it may be useful, by way of motivation, to first peruse the proof of Lemma 2.6. The basic intuition is that the $a_i$ represent level sizes of a normalized matching poset and the $x_i$ the sizes of subsets of each level, and we are interested in a lower bound for the size of the set of neighbors of these subsets, with the restriction that we only consider neighbors which are above, but not too far above, each element, where 'too far above' is as delimited by $f$.

It may also be helpful to think of $f$ just in terms of sequences of integers. This function, given an input $k$, returns the last index in the smallest contiguous block of entries immediately after index $k$ with values summing to at least the value of the largest entry $a_0$ in the sequence.

**Proof** We proceed by induction on $p$.

**Base Case:** Suppose $p = 0$. Then, since $\{0, \ldots, p\} = \{0\}$ and $f(0) = m$,

$$\sum_{i=1}^{m} a_i \max\left\{ \frac{x_k}{a_k} \;\middle|\; k \in \{0, \ldots, p\}, k < i \leq f(k) \right\}$$

is equal to

$$\sum_{i=1}^{m} a_i \max\left\{ \frac{x_0}{a_0} \;\middle|\; 0 < i \leq m \right\} = \sum_{i=1}^{m} a_i \frac{x_0}{a_0} = \frac{x_0}{a_0} \sum_{i=1}^{f(p)} a_i,$$

which is, by the definition of $f$, at least

$$\frac{x_0}{a_0} a_0 = x_0 = \sum_{i=0}^{p} x_i.$$

The result follows.

**Hypothesis of Induction:** Let $\mathfrak{p} \in \mathbb{N}$ and suppose that the result holds for all such sequences with $p = \mathfrak{p} - 1$ to show that it holds whenever $p = \mathfrak{p}$.

**Inductive Step:** Let $a_0 \geq \ldots \geq a_m$ be such a sequence with $p = \mathfrak{p}$. Define $s_i$ for $i \in \{0, \ldots, m\}$ recursively by setting $s_i = \min\{a_i, a_\mathfrak{p} - \sum_{j=i+1}^{m} s_j\}$ and let $a'_i = a_i - s_i$ for all $i \in \{0, \ldots, m\}$. Then we can see that the sum of the sequence $(a'_i)$ is exactly $a_\mathfrak{p}$ less than that of $(a_i)$, and we have accomplished this by removing as much value as possible from the highest-indexed entries in our sequence. Let $\ell$ be the largest value such that $a'_\ell$ is greater than 0.

Let $f'$ be $(a'_i)$'s equivalent to $f$, where we view $(a'_i)$ as a sequence of positive integers by restricting $i$ to $\{0, \ldots, \ell\}$. Then we can see that $f'(\mathfrak{p} - 1)$ is finite. Because $a_{\mathfrak{p}+1} + \ldots + a_m \geq a_0 \geq a_\mathfrak{p}$ by the definition of $p = \mathfrak{p}$ and we constructed the primed sequence by reducing higher-indexed values first, we can see that $s_\mathfrak{p}$ is equal to zero and so $a'_i = a_i$ for all $i \leq \mathfrak{p}$. Therefore, using the fact that $a'_i = 0$ for $i > \ell$, we have

$$a'_\mathfrak{p} + \ldots + a'_\ell = a'_\mathfrak{p} + \ldots + a'_m = a_\mathfrak{p} + (a_{\mathfrak{p}+1} + \ldots + a_m - a_\mathfrak{p}) = a_{\mathfrak{p}+1} + \ldots + a_m,$$

which is again greater than or equal to $a_0 = a'_0$ by the definition of $p$. The claim of finiteness follows.

Moreover, $f'(\mathfrak{p})$ is infinite, since we have

$$a'_{\mathfrak{p}+1} + \ldots + a'_\ell = a_{\mathfrak{p}+1} + \ldots + a_m - a_\mathfrak{p} \leq a_{\mathfrak{p}+1} + \ldots + a_{m-1}$$

and so $a'_{\mathfrak{p}+1} + \ldots + a'_\ell \geq a_0$ would contradict the hypothesis $f(\mathfrak{p}) = m$.

Let $m' = f'(\mathfrak{p} - 1) \leq \ell$, and truncate $(a'_i)$ to a sequence with highest index $m'$. Then we can see that the analogue to $f$ for this sequence will agree with $f'$ wherever both are defined, since the terms of $(a'_i)$ with indices larger than $m'$ will never affect the calculation of $f'(k)$ for $k \leq \mathfrak{p} - 1$, due to the fact that $a'_{k+1} + \ldots + a'_i \leq a_{\mathfrak{p}'} + \ldots + a'_i$ always for such $k$, and both functions will be infinite on all $\mathfrak{p} \leq k \leq m'$.

Moreover, the $p$-value of this new sequence will be $\mathfrak{p} - 1$. Thus, by the hypothesis of induction, we have

$$\sum_{i=1}^{m'} a'_i \max\left\{ \frac{x_k}{a'_k} \;\middle|\; k \in \{0, \ldots, \mathfrak{p} - 1\}, k < i \leq f'(k) \right\} \geq \sum_{i=0}^{\mathfrak{p}-1} x_i,$$

where we need not modify the $x_k$ since $a'_k = a_k$ for the $k$ we are considering. The expression on the left-hand side of this inequality is equal to

$$\sum_{i=1}^{m'} (a_i - s_i) \max \left\{ \frac{x_k}{a_k - 0} \;\middle|\; k \in \{0, \dots, \mathfrak{p} - 1\}, k < i \leq f'(k) \right\}.$$

Fix $k \leq \mathfrak{p} - 1$. Note that $f'(k) \leq \ell$ and hence $a'_i = a_i$ for $i < f'(k)$. As such,

$$a_{k+1} + \ldots + a_{f'(k)-1} = a'_{k+1} + \ldots + a'_{f'(k)-1} < a'_0 = a_0,$$

so $f(k) \geq f'(k)$. Moreover, since $a_{f'(k)} \geq a'_{f'(k)}$ by definition, we have

$$a_{k+1} + \ldots + a_{f'(k)} \geq a'_{k+1} + \ldots + a'_{f'(k)} \geq a'_0 = a_0,$$

so in fact $f(k) = f'(k)$.

As such, by the non-negativity of the values involved, we can add some terms and expand the set which we are taking the maximum of slightly to see that our sum is bounded above by

$$\sum_{i=1}^{m} (a_i - s_i) \max \left\{ \frac{x_k}{a_k} \;\middle|\; k \in \{0, \dots, \mathfrak{p}\}, k < i \leq f(k) \right\},$$

which is equal to

$$\sum_{i=1}^{m} a_i \max \left\{ \frac{x_k}{a_k} \;\middle|\; k \in \{0, \dots, \mathfrak{p}\}, k < i \leq f(k) \right\}$$

minus

$$\sum_{i=\ell}^{m} s_i \max \left\{ \frac{x_k}{a_k} \;\middle|\; k \in \{0, \dots, \mathfrak{p}\}, k < i \leq f(k) \right\}.$$

Since every index $\ell$ or above is greater than $\mathfrak{p}$ but less than or equal to $f(\mathfrak{p}) = m$, the subtrahend is greater than or equal to

$$\sum_{i=\ell}^{m} s_i \frac{x_{\mathfrak{p}}}{a_{\mathfrak{p}}} = \frac{x_{\mathfrak{p}}}{a_{\mathfrak{p}}} \sum_{i=\ell}^{m} s_i = \frac{x_{\mathfrak{p}}}{a_{\mathfrak{p}}} a_{\mathfrak{p}} = x_{\mathfrak{p}}.$$

Therefore the left-hand side of our original inequality is bounded above by

$$\left(\sum_{i=1}^{m} a_i \max\left\{\frac{x_k}{a_k} \;\middle|\; k \in \{0,\ldots,\mathfrak{p}\}, k < i \le f(k)\right\}\right) - x_{\mathfrak{p}};$$

since the right-hand side is equal to $\sum_{i=0}^{\mathfrak{p}} x_i - x_{\mathfrak{p}}$, adding $x_{\mathfrak{p}}$ to both sides of the resulting inequality gives us

$$\sum_{i=1}^{m} a_i \max\left\{\frac{x_k}{a_k} \;\middle|\; k \in \{0,\ldots,\mathfrak{p}\}, k < i \le f(k)\right\} \ge \sum_{i=0}^{\mathfrak{p}} x_i,$$

as desired.

The induction proves the claim. ∎

We can now proceed to the following lemma on monotone normalized matching posets.

**Lemma 2.6 ([21])** *Let $P$ be a normalized matching poset of width $w$ with levels $A_0, A_1, \ldots, A_n$ and monotonically non-increasing rank numbers, so that $w = |A_0|$. Define a function $f : \{0, 1, \ldots, n\} \cup \{\infty\} \to \{0, \ldots, n\} \cup \{\infty\}$ by*

$$f(k) = \begin{cases} \min\{k < i \le n \mid |A_{k+1}| + \ldots + |A_i| \ge w\} & \textit{if this set is non-empty} \\ \infty & \textit{otherwise} \end{cases}$$

*and let $d$ be the largest integer such that $f^d(0) < \infty$. Then $P$ can be partitioned into $w$ chains, each of size at least $d + 1$.*

Although the statement of this result is far from intuitive, the basic concept of the proof is surprisingly natural. We will build a chain partition of $P$ from the bottom up; that is, we start with an element of $A_0$ and construct a chain by repeatedly adding in an element greater than the current top of our chain. However, we do so myopically, without being able to look very far upward: the levels from which we are allowed to select the element we are to add are delimited by the result of $f$ applied to the rank of current top element. Hence, since $d$ then gives us some lower bound on the number of steps we must take to reach the top of the poset and be forced to terminate, we can get a corresponding lower bound on the size of our chains.

**Proof** Let $p$ be the largest element of $\{0, \ldots, n\}$ such that $f(p)$ is finite, and note that, although their definitions are similar, $p$ and $d$ are not, in general, equal. Also note that $p < n$, since we can see that $f(n) = \infty$.

Now let $A$ be an unordered copy of the levels $A_0, \ldots, A_p$ of $P$; that is, for every $a \in A$, there is a unique associated element $\phi_A(a)$ in $\bigcup_{i=0}^{p} A_i$, and this makes $\phi_A$ a bijection. Similarly, let $B$ be an unordered copy of the levels $A_1, \ldots, A_{f(p)}$ with bijection $\phi_B : B \to \bigcup_{i=1}^{f(p)} A_i$.

Having done this, define a bipartite graph $G = (A, \Delta, B)$ by $(a, b) \in \Delta \Leftrightarrow \big(\mathrm{rk}(\phi_B(b)) \leq f(\mathrm{rk}(\phi_A(a)))$ and $\phi_A(a) < \phi_B(b)\big)$. In practical terms, this means that the edges of our bipartite graph are given by relationships in $P$, but only those such that the element of $A$ we are considering is strictly less than the element of $B$ and the element of $B$ is within the block delimited by $f$ applied to the element of $A$. This restriction corresponds to the 'myopia' which we discussed earlier.

We will now show that it suffices to produce a matching in $G$ which covers $A$. Suppose that we have such a matching, and hence that a maximum matching in $G$ has $|A|$ edges. Now observe for any $0 \leq i < n$ that we have a matching in $P$ between $A_i$ and $A_{i+1}$ which covers $A_{i+1}$ by Proposition 1.36, since $P$ is normalized matching and $|A_{i+1}| \leq |A_i|$. Since all the edges in these matchings are preserved in $G$ for $0 \leq i \leq p$, we can use this fact to construct a matching in $G$ between $A = {\phi_A}^{-1}(\bigcup_{i=0}^{p} A_i)$ and the newly defined set $B_0 = {\phi_B}^{-1}(\bigcup_{i=1}^{p+1} A_i)$ which covers $B_0$. By Theorem 1.31, there then exists a maximum matching in $G$ which covers $B_0$; since a maximum matching in $G$ has $|A|$ edges and $G$ is bipartite, this gives us a matching $M$ in $G$ which covers both $A$ and $B_0$.

We use this matching to construct pairwise disjoint chains $\{C_x\}_{x \in A_0}$ in $P$ as follows. In a slight abuse of notation, we will think of $M$ as an injective function $M : A \to B$ which takes each element of $A$ to its matched element of $B$. Then our process will be to begin with $C_x = \{x\}$ and iteratively add the element $(\phi_B \circ M \circ {\phi_A}^{-1})(y)$ of $P$ which $M$ matches to $y$'s counterpart in $A$, where $y$ is the current maximal element in $C_x$, to $C_x$. We will stop after the first time we add a point with rank greater than $p$ to $C_x$. Put more simply, the chains $C_x$ which we obtain are the ones which result from following all edges in $M$ while identifying elements of $A$ and $B$ which correspond to the same element of $P$ with one another.

We claim that, for each $x \in A_0$, $|C_x| \geq d + 1$. Recall by the construction of $G$ that, if $y < z$ are consecutive elements of $C_x$, $\mathrm{rk}(z) \leq f(\mathrm{rk}(y))$. Since we can see from the definition that $f$ is monotonically increasing, a simple inductive argument tells us that the $\ell$th-smallest element of $C_x$ has rank at most $f^{\ell-1}(0)$. The largest element of $C_x$ has rank at least $p+1$, so this means $f^{|C_x|-1}(0) \geq p+1$. By the definition of $p$, this means that $f^{|C_x|}(0)$ is infinite, so $|C_x| \geq d+1$ as desired.

We now extend these chains to cover all of $P$, giving us the partition we sought. For each $i \in \{p+2, \ldots, n\}$, let $N_i$ be a matching from $A_i$ to $A_{i-1}$ which covers $A_i$; continuing our abuse of notation, we will again think of this a function $N_i : A_i \to A_{i-1}$. Note that, since $M$ covered $A$ and $B_0$, the chains $C_x$ collectively cover the levels $A_0, \ldots, A_{p+1}$. Therefore, we can add the elements of $P$ not covered by the $C_x$ to them as follows:

- At each stage, let $y$ be an element of minimal rank among the elements not yet in chains.

- Let $z = N_{\mathrm{rk}(y)}(y)$ be the element to which $y$ is matched by the appropriate matching from among the $N_i$. Observe by the minimality of $y$ that $z \in C_x$ for some $x \in A_0$.

- Moreover, note that $z$ is maximal in $C_x$. If $z$ was added to $C_x$ during its original construction, it was the last element thus appended, since we stopped building $C_x$ as soon as we included an element of rank more than $p$; since $\mathrm{rk}(y) \geq p+2$, $\mathrm{rk}(z) \geq p+1$. If $z$ was added to $C_x$ during this process, the minimality of $y$ and the injectivity of $N_{\mathrm{rk}(y)}$ guarantee that no elements larger than $z$ have yet been added to $C_x$. Therefore, add $y$ to $C_x$; the result will still be a chain.

- Repeat these steps until the chains cover all elements of $P$.

Therefore, the lemma holds so long as we can actually produce a matching on $G$ which covers $A$. By the Marriage Theorem (Theorem 1.33), it suffices to show that, if $\Gamma(X)$ denotes the set of neighbors of elements of $X \subseteq A$, $|\Gamma(X)| \geq |X|$ for all such $X$.

Consider arbitrary such $X$ and, for each $i \in \{0, \ldots, p\}$, let $X_i = X \cap {\phi_A}^{-1}(A_i)$; that is, $X_i$ is the set of elements of $X$ corresponding to elements on level $A_i$ in $P$. Similarly, for $j \in \{1, \ldots, f(p)\}$, let $\Gamma_j : \mathcal{P}(A) \to \mathcal{P}(B)$ be given by $\Gamma_j(Z) = \Gamma(Z) \cap {\phi_B}^{-1}(A_j)$; this is the set of neighbors of $Z$ corresponding to elements on level $A_j$ in $P$.

Then we can see that $\Gamma(X) = \bigcup_{k=0}^{p} \bigcup_{i=k+1}^{f(k)} \Gamma_i(X_k)$. Interchanging the unions appropriately and using the disjointness of the $\Gamma_i$s' images gives us

$$|\Gamma(X)| = \sum_{i=1}^{f(p)} \left| \bigcup_{k\in\{0,\ldots,p\},k<i\leq f(k)} \Gamma_i(X_k) \right| \geq \sum_{i=1}^{f(p)} \max_{k\in\{0,\ldots,p\},k<i\leq f(k)} |\Gamma_i(X_k)|.$$

Since $P$ is normalized matching, this in turn is greater than or equal to

$$\sum_{i=1}^{f(p)} \max_{k\in\{0,\ldots,p\},k<i\leq f(k)} |A_i| \frac{|X_k|}{|A_k|}.$$

However, we can see that, if we let $m = f(p)$ and $a_i = |A_i|$ and $x_i = |X_i|$ for all appropriate $i$, then our function $f$ is actually the function $f$ from the previous lemma, and the previous expression is one side of the inequality in that lemma's result, so we can apply it to obtain $|\Gamma(X)| \geq \sum_{i=0}^{p} |X_i| = |X|$ as desired. As previously discussed, the result follows. ∎

Although this lemma is quite powerful, it is desirable to state our bounds in terms of parameters of our poset more commonly used than $d$. This goal gives rise to the following bound for $d$, which will be useful in our proof of Theorem 2.4.

**Lemma 2.7 ([21])** *Suppose that the hypotheses of Lemma 2.6 hold with $w > 1$. Then $d + 1 \geq \frac{|P|-1}{2(w-1)}$.*

**Proof** We claim that, if we take $A_i$ to be the empty set for all $i > n$, we have $\sum_{i=k+1}^{f(k)} |A_i| \leq 2w - 2$ for all $k \in \{0, 1, \ldots, n\}$. In the case where $f(k)$ is finite and $|A_{f(k)}|$ is strictly less than $w$, this is because $\sum_{i=k+1}^{f(k)} |A_i| \geq 2w - 1$ would imply that $\sum_{i=k+1}^{f(k)-1} |A_i| > w - 1$ and so contradict the minimality of $f(k)$. If $|A_{f(k)}| = w$, we have $f(k) = k + 1$ and so our sum is equal to $w$, which is bounded above by $2w - 2$ by the hypothesis $w > 1$. Finally, if $f(k)$ is infinite, the definition of $f$ tells us that the sum of the sizes of all levels above $k$ is less than $w \leq 2w - 2$, so the inequality holds in all cases.

Thus, for any $\ell \geq 0$, we can see that

$$\sum_{i=0}^{f^{\ell}(0)} |A_i| = w + \sum_{j=1}^{\ell} \sum_{i=f^{j-1}(0)+1}^{f^j(0)} |A_i| \leq w + \ell(2w - 2),$$

so the inequality $|P| - w - \ell(2w-2) \geq w$ implies that there are at least $w$ elements above level $f^{\ell}(0)$ and hence that $d > \ell$. Therefore, we have $|P| - w - d(2w-2) \leq w - 1$, which gives us $d \geq \frac{|P|-2w+1}{2(w-1)} = \frac{|P|-1}{2(w-1)} - 1$. ∎

We are now in a position to prove the result which was our original goal without too much additional work. As in the proof of Theorem 2.2, we break our unimodal poset into two overlapping monotone posets, then apply the appropriate lemma and combine the resulting chain partitions.

**Proof of Theorem 2.4** Let $P$ a unimodal normalized matching poset of width $w$ with levels $A_0, A_1, \ldots, A_n$, and let $\ell \in \{0, \ldots, n\}$ be such that $|A_\ell| = w$. If we let $U$ be the induced subposet of $P$ on all levels of index at least $\ell$ and $L$ on all levels of index at most $\ell$, we can observe as in the proof of Theorem 2.2 that $|U| + |L| = |P| + w$.

Using Lemmas 2.6 and 2.7 together, we can obtain partitions of these posets into $w$ each chains with sizes at least $\frac{|U|-1}{2(w-1)}$ and $\frac{|L|-1}{2(w-1)}$ respectively. Combining these partitions by merging on the chains' intersections with $A_\ell$, as in the proof of Theorem 2.2, yields a partition into $w$ chains with sizes at least $\frac{|P|+w-2}{2(w-1)} - 1 = \frac{|P|-1}{2(w-1)} - \frac{1}{2}$. ∎

In practice, it may be convenient to use a slightly weaker but less cumbersome version of this bound, as originally presented in [21].

**Corollary 2.8** *Let $P$ be a unimodal normalized matching poset of width $w$. Then $P$ can be partitioned into $w$ chains of size at least $\frac{|P|}{2w} - \frac{1}{2}$.*

**Proof** In the $w > 1$ case, this follows by arithmetic from Theorem 2.2 and the inequality $w \leq |P|$. If $w = 1$, $P$ is totally ordered and hence the result is trivial. ∎

### 2.1.3 A probabilistic method for obtaining chain partitions

Theorems 2.2 and 2.4 give fairly decent bounds in the case of unimodal normalized matching posets. However, it is clear that they are more or less useless if we wish to answer the Füredi question for a general, not necessarily unimodal, normalized matching poset, since the methodology we used to prove them depended heavily on the decomposition into two monotone subposets. To give some idea of what approach we might use in the general case, we present a probabilistic technique for obtaining a chain partition with sizes bounded below developed by Tomon [21].

To state the result, we require a bit of specialized notation. Basic calculus tells us that, when restricted to the positive real numbers, $x \mapsto xe^x$ is strictly increasing and therefore invertible.

**Definition 2.9** We define $W : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of positive real numbers, to be the inverse of the function $x \mapsto xe^x$.

In the study of complex analysis, the **product logarithm** $W$ refers to a *family* of functions analogous to the multiple branches of the complex logarithm, since the function $z \mapsto ze^z$ from the complex plane to itself is not invertible. Since, as we have noted, no such complications arise in the positive real case, we need not concern ourselves with the particulars. The only properties of $W$ which we will use are its definition as an inverse and a numerical approximation, determined computationally, for its output at $1/e$. We also prove the following simple lemma, which will come in handy later.

**Lemma 2.10** *Let $\kappa \in \mathbb{R}$ such that $\kappa \geq \frac{1}{W(1/e)} \approx 3.591121476669$. Then $\kappa(1 - \log \kappa) \leq -1$.*

**Proof** Observe that $\frac{d}{d\kappa}\kappa(1 - \log \kappa) = 1 - \log \kappa - 1 = -\log \kappa$. By the bound on $\kappa$, this will be negative, so it suffices to demonstrate the result for $\kappa = \frac{1}{W(1/e)}$. In this case, we have $e = (1/e)^{-1} = (\frac{1}{\kappa}e^{1/\kappa})^{-1} = \kappa e^{-1/\kappa}$. Therefore, $1 - \log \kappa = \log(e/\kappa) = -\frac{1}{\kappa}$, so $\kappa(1 - \log \kappa) = -1$. ∎

With these preliminaries accounted for, we are now prepared to examine the statement of Tomon's result.

**Theorem 2.11 ([21])** *Let $P$ be a normalized matching poset of rank $n$ and width $w$ containing at least two elements and consider arbitrary $\kappa \geq \frac{1}{W(1/e)} \approx 3.591121476669$. Then, if*

$$n + 1 \leq \frac{|P|^2}{2\kappa^2 w^2 (\log(2|P|))^2 \log \lceil w \log(2|P|) \rceil},$$

*there exists a partition of $P$ into no more than $\lceil w \log(2|P|) \rceil$ chains such that each chain has size greater than $\frac{|P|}{2\kappa w \log(2|P|)}$.*

Note that, although we have dropped the unimodality requirement, the conditions needed for this theorem to apply are not necessarily less restrictive; for example, the Boolean lattices do not fulfill the necessary inequality. Moreover, in at least some unimodal cases where this theorem also applies, Theorem 2.4 appears to give better bounds [21], and in all cases the partition which we produce is not guaranteed to have the minimum number of chains. In spite of these drawbacks, this result is notable for the methodology used, which is unorthodox and worthy of further exploration.

We make use of the following concept, in some ways similar to that of a chain partition and in some ways very different.

**Definition 2.12** Let $P$ be a ranked poset. Then a **regular cover** of $P$ is a finite collection $C_1, \ldots, C_r$ of maximal chains of $P$ (which are not necessarily disjoint or distinct) such that each element of $P$ appears on the same positive number of chains in the collection as every other element of the same rank.

A theorem of Kleitman describes the circumstances under which it is possible to obtain a regular cover for a poset.

**Theorem 2.13 ([16])** *Let $P$ be a ranked poset. Then $P$ has a regular cover if and only if it is normalized matching.*

Thus the existence of a regular cover, like the LYM property, turns out to be equivalent to the normalized matching property.

One further technical result from the study of probability is needed.

**Lemma 2.14 ([11])** *Let $X_1, \ldots, X_n$ be independent random variables with values in $\{0, 1\}$ and $X$ their sum. Then, for any $t > 0$, the probability that $\mathbb{E}(X) - X \geq t$, where $\mathbb{E}(X)$ denotes the expected value of $X$, is strictly less than $e^{-2t^2/n}$.*

We are now ready to prove Theorem 2.11. The basic concept of the proof is that we choose some number of maximal chains randomly from some fixed regular cover of our poset, then demonstrate that there is a nonzero probability that these chains have the properties necessary for us to use them to construct a chain partition of the poset with bounded chain sizes. Since a nonzero probability means that there is some choice of chains satisfying our criteria, the result will follow.

**Proof of Theorem 2.11** Fix some regular cover $\mathcal{C}$ of $P$ and choose $M = \lceil w \log(2|P|) \rceil$ chains, not necessarily distinct, randomly and independently from $\mathcal{C}$, where the probability of picking any given chain is simply the number of times it appears in $\mathcal{C}$ as a fraction of the total number of chains, counting repeats, in the cover. Denote these chains $C_1, \ldots, C_M$.

Observe that, if $M \geq |P|$, we can partition $P$ into $M$ singletons, which will satisfy our requirement that the chains of the partition have size at least $\frac{|P|}{2\kappa w \log(2|P|)} < \frac{|P|}{2\kappa(M-1)} = \frac{|P|-1}{2\kappa(M-1)} + \frac{1}{2\kappa(M-1)} \leq \frac{1}{\kappa} < 1$. Therefore, we may assume without loss of generality that $M < |P|$.

Now, for every $x \in P$, define $n_x$ to be the number of elements $i \in \{1, \ldots, M\}$ such that $x \in C_i$, and let $\mathcal{A}$ denote the event that $n_x > 0$ for every $x \in P$. To establish a lower bound on the probability that $\mathcal{A}$ occurs, we consider the probability $\mathbb{P}(\overline{\mathcal{A}}) = \mathbb{P}(\exists x \in P \mid x \notin C_i \quad \forall i \in [M])$ that it does not occur. We can obtain an overestimate of this quantity by taking the sum $\sum_{x \in P} \mathbb{P}(x \notin C_i \quad \forall i \in [M])$ of the probabilities of $x$ being left out over all elements $x$ and ignoring the intersections of these events. Now note that, for any element $x$ and chain $C_i$, $C_i$ contains some element on the level of $x$ and, by the definition of a regular cover, this is exactly as likely to be $x$ as it is any other element on the level. Therefore our sum becomes $\sum_{x \in P}(1 - 1/|A_{\mathrm{rk}(x)}|)^M \leq |P|(1 - 1/w)^M$.

In order to obtain a more convenient approximation, we note that

$$\begin{aligned} e^{-1/w} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k! w^k} \\ &= 1 - \frac{1}{w} + \frac{1}{2w^2} - \frac{1}{6w^3} + \ldots \\ &= 1 - \frac{1}{w} + \sum_{k=1}^{\infty} \left( \frac{1}{(2k)! w^{2k}} - \frac{1}{(2k+1)! w^{2k+1}} \right) \\ &= 1 - \frac{1}{w} + \sum_{k=1}^{\infty} \frac{(2k+1)w - 1}{(2k+1)! w^{2k+1}} > 1 - \frac{1}{w}, \end{aligned}$$

so $\mathbb{P}(\overline{\mathcal{A}}) < |P| e^{-M/w} \leq |P| e^{-\log(2|P|)} = \frac{|P|}{2|P|} = \frac{1}{2}$. As such, $\mathbb{P}(\mathcal{A}) > \frac{1}{2}$.

Now, for each $i \in \{0, \ldots, n\}$, let $L_i = \lceil \kappa w \log(2|P|)/|A_i| \rceil$. Denote by $\mathcal{B}$ the event that $n_x < L_{\mathrm{rk}(x)}$ for all $x \in P$ and consider $\mathbb{P}(\overline{\mathcal{B}})$. In particular, observe that we can, as before, overestimate this probability by ignoring the intersections of events, giving us

$$\sum_{i=0}^{n} \sum_{x \in P} \mathbb{P}(\exists \ell_1, \ldots, \ell_{L_i} \in [M] \text{ distinct} \mid x \in C_{\ell_j} \quad \forall j \in [L_i])$$

as an upper bound. Again overestimating by summing over all possible choices of the $\ell_j$, we obtain the upper bound $\sum_{i=0}^{n} |A_i| \left(\frac{1}{|A_i|}\right)^{L_i} \binom{M}{L_i}$.

Now note that, if we let $\mathfrak{m} = w \log(2|P|)$, we have $M < \mathfrak{m} + 1$ and so $M(M-1)\dots(M-L_i+1) < (\mathfrak{m}+1)\mathfrak{m}\dots(\mathfrak{m}-L_i+2)$, which, since $(\mathfrak{m}+1)(\mathfrak{m}-1) = \mathfrak{m}^2 - 1 < \mathfrak{m}^2$, is less than $\mathfrak{m}^{L_i}$ whenever $L_i \geq 3$. Therefore, in these cases, we have

$$\binom{M}{L_i} < \frac{\mathfrak{m}^{L_i}}{L_i!} = (e\mathfrak{m})^{L_i} \frac{1}{L_i! e^{L_i}} = (e\mathfrak{m})^{L_i} \left(L_i! \sum_{k=0}^{\infty} \frac{{L_i}^k}{k!}\right)^{-1} < (e\mathfrak{m})^{L_i} \left({L_i}^{L_i}\right)^{-1}.$$

If $L_i = 1$, we can see that $\binom{M}{L_i} = M < \mathfrak{m} + 1 < e\mathfrak{m} = (e\mathfrak{m}/L_i)^{L_i}$ unless $\mathfrak{m} \leq \frac{1}{e-1}$, which is impossible since $1 \leq w$ and $2 \leq |P|$. If $L_i = 2$, we have $\binom{M}{L_i} = \frac{M(M-1)}{2} < \frac{\mathfrak{m}(\mathfrak{m}+1)}{2}$, which is less than $(e\mathfrak{m}/2)^2$ unless $\mathfrak{m} \leq \frac{2}{e^2-2}$; this again contradicts our known bounds for $\mathfrak{m}$. Thus $\binom{M}{L_i} < \left(\frac{e\mathfrak{m}}{L_i}\right)^{L_i}$ in all cases.

Therefore $\mathbb{P}(\overline{\mathcal{B}}) < \sum_{i=0}^{n} |A_i| \left(\frac{e\mathfrak{m}}{|A_i| L_i}\right)^{L_i} \leq \sum_{i=0}^{n} |A_i| \left(\frac{ew \log(2|P|)}{\kappa w \log(2|P|)}\right)^{L_i}$, which is equal to $\sum_{i=0}^{n} |A_i| \left(\frac{e}{\kappa}\right)^{L_i}$. Since $\kappa > e$, this in turn is less than or equal to $\sum_{i=0}^{n} |A_i| \left(\frac{e}{\kappa}\right)^{\kappa w \log(2|P|)/|A_i|} \leq \sum_{i=0}^{n} |A_i| \left(\frac{e}{\kappa}\right)^{\kappa \log(2|P|)} = |P| \left(\frac{e}{\kappa}\right)^{\kappa \log(2|P|)}$. Now note that $\left(\frac{e}{\kappa}\right)^{\kappa \log(2|P|)} = e^{\kappa \log(2|P|) \log(e/\kappa)} = (2|P|)^{\kappa(1-\log \kappa)}$. By Lemma 2.10, this is at most $(2|P|)^{-1}$, so $\mathbb{P}(\overline{\mathcal{B}}) < \frac{1}{2}$ and hence $\mathbb{P}(\mathcal{B}) > \frac{1}{2}$.

Therefore, since $\mathcal{A}$ and $\mathcal{B}$ each occur in over half of all cases, $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) > 0$, and hence there exists a choice $C_1, \dots C_M$ of chains such that $0 < n_x < L_{\mathrm{rk}(x)}$ for each $x \in P$. Henceforth, suppose that we have picked such a collection of chains.

At this point we do not have a chain partition of $P$, since our chains may overlap wholly or partially. To remedy this, we apportion elements at random to the chains we have chosen. That is, we assign each $x$ a random index $\iota_x$, chosen with uniform probabilities from the set $\{i \in [M] \mid x \in C_i\}$ of indices of chains containing $x$. For each $i \in [M]$, let $D_i = \{x \in C_i \mid \iota_x = i\}$ be the set of elements assigned to index $i$. Then we can see that $D_1, \dots, D_M$ will be a chain partition of $P$, since each element is in exactly one of the $D_i$ and each $D_i$ is a chain. Note that some of the $D_i$ may be empty, in which case we will have a partition into fewer than $M$ chains.

Now we show that there is a nonzero probability of these chains' sizes having the desired lower bound. Let $t = \frac{|P|}{2\kappa w \log(2|P|)}$ and observe that, since each $x$ has a $\frac{1}{n_x}$ chance of ending up on any of its possible chains,

$$\mathbb{E}(|D_i|) = \sum_{x \in C_i} \mathbb{P}(\iota_x = i) = \sum_{x \in C_i} \frac{1}{n_x} \geq \sum_{j=0}^{n} \frac{1}{L_j - 1} > \sum_{j=0}^{n} \frac{|A_j|}{\kappa w \log(2|P|)} = 2t$$

for each $i \in [M]$. Therefore $\mathbb{P}(|D_i| \leq t) \leq \mathbb{P}(\mathbb{E}(|D_i|) - |D_i| \geq t)$. However, we can view $D_i$ as the sum of the indicator variables for the independent events $\iota_x = i$ for all $x \in C_i$, so by Lemma 2.14 this latter probability is strictly less than $e^{-2t^2/|C_x|} = e^{-2t^2/(n+1)}$. By our bound for $n+1$, this is at most $e^{-\log M} = \frac{1}{M}$.

Therefore, $\mathbb{P}(\exists i \in [M] \mid |D_i| \leq t) \leq \sum_{i=1}^{M} \mathbb{P}(D_i \leq t) < \frac{M}{M} = 1$, so the probability that our partition does have chains sizes greater than $t$ is nonzero. Hence there is some partition that fulfills our criteria. ∎

This probabilistic mode of reasoning provides an interesting alternative to the more concrete methods of constructing chain partitions which we have been using thus far. Unfortunately, as we have discussed, this particular result is not of much use for our purposes, so we focus on applying the others to the Boolean lattice.

## 2.2 Results on the Boolean lattice

Theorems 2.2 and 2.4, since they apply to all unimodal normalized matching posets, are valid for the Boolean lattice, and indeed we could apply them directly to find explicit formulae for their bounds in the case of $2^{[n]}$ if we so chose. However, as we have mentioned, obtaining a partition into chains whose sizes have an upper bound only or a lower bound only will never be sufficient to prove the Füredi conjecture. As such, it is desirable to obtain chain partitions with sizes bounded both above and below.

### 2.2.1 An asymptotic view of the Füredi partition

In particular, we are interested in the long-term behavior of any method for obtaining such a partition. This is to say that, since we can search for Füredi partitions of any finite number of Boolean lattices if need be, we would really like to know how our approach behaves on $2^{[n]}$ as $n$ becomes infinitely large. Therefore, we turn to an asymptotic analysis. The following estimate is well-known and will be used without proof.

**Lemma 2.15 (Stirling's Approximation)** *The factorial function may be approximated by*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Here, $\sim$ denotes asymptotic equivalence; that is, for nonzero sequences $(a_n)$ and $(b_n)$, we say that $a_n \sim b_n$ when the ratio $\frac{a_n}{b_n}$ tends to 1 as $n \to \infty$. Stirling's approximation can be derived in several ways, none of which we will pursue in detail. However, we note that one possible proof involves the following definition from complex analysis.

**Definition 2.16** We define the **gamma function** $\Gamma$ as the analytic continuation of the function

$$s \mapsto \int_0^\infty e^{-t} t^{s-1}\, dt$$

defined on the open half-plane of complex numbers with positive real part to the entire complex plane, excluding the nonpositive integers.

The reader unfamiliar with such terminology need not be unduly concerned by it; for the curious, Stein and Shakarchi's *Complex Analysis* [20] gives a good introduction to the topic. Our interest in the gamma function arises from the following well-known fact.

**Lemma 2.17** *Let $n$ be a positive integer. Then $\Gamma(n) = (n-1)!$.*

The preceding identity can be used to deduce Stirling's approximation, which, indeed, can be applied to the gamma function more generally. This, in turn, allows us to get a more tractable asymptotic formula for the width of the Boolean lattice.

**Lemma 2.18** *The width $\binom{n}{\lfloor n/2 \rfloor}$ of the Boolean lattice $2^{[n]}$ is asymptotically equivalent to $2^n\sqrt{\frac{2}{\pi n}}$.*

**Proof** We have

$$\binom{n}{\lfloor n/2 \rfloor} = \frac{n!}{\lfloor n/2 \rfloor! \lceil n/2 \rceil!} = \frac{n!}{\frac{n-c_n}{2}!\frac{n+c_n}{2}!},$$

where $c_n$ is 0 if $n$ is even and 1 if $n$ is odd. By Stirling's approximation and some algebra, this is asymptotically equivalent to

$$\frac{\sqrt{2\pi n}(n/e)^n}{\pi\sqrt{n^2-{c_n}^2}\left(\frac{n-c_n}{2e}\right)^{\frac{n-c_n}{2}}\left(\frac{n+c_n}{2e}\right)^{\frac{n+c_n}{2}}} = \frac{\sqrt{2\pi n}(n/e)^n}{\pi\sqrt{n^2-{c_n}^2}\left(\frac{n^2-{c_n}^2}{4e^2}\right)^{\frac{n}{2}}\left(\frac{n+c_n}{n-c_n}\right)^{\frac{c_n}{2}}},$$

which in turn is equal to

$$2^n\sqrt{\frac{2n}{\pi(n^2-{c_n}^2)}\cdot\left(\frac{n-c_n}{n+c_n}\right)^{c_n}}\left(\frac{n}{\sqrt{n^2-{c_n}^2}}\right)^n.$$

In the cases where $c_n = 0$, this evaluates to $2^n\sqrt{\frac{2}{\pi n}}$ as desired. Otherwise, it is equal to

$$2^n\sqrt{\frac{2n}{\pi(n^2-1)}\cdot\frac{n-1}{n+1}}\left(\frac{n}{\sqrt{n^2-1}}\right)^n = 2^n\sqrt{\frac{2n}{\pi(n+1)^2}}\left(\frac{n}{\sqrt{n^2-1}}\right)^n.$$

Dividing by $2^n\sqrt{\frac{2}{\pi n}}$ to show that the result converges to 1 and hence that the two expressions are asymptotically equivalent, we obtain

$$\sqrt{\frac{n^2}{(n+1)^2}}\left(\frac{n}{\sqrt{n^2-1}}\right)^n = \frac{n}{n+1}\left(\frac{n}{\sqrt{n^2-1}}\right)^n.$$

Since $\frac{n}{n+1}$ converges to 1, it suffices to show that $\left(\frac{n}{\sqrt{n^2-1}}\right)^n$ does as well. This is true, although the calculations are tedious and we do not reproduce them here.

∎

This can be used to get an asymptotic approximation for the sizes of the chains in a Füredi partition.

**Corollary 2.19** *The sizes of the chains in the Füredi partitions of $2^{[n]}$ are asymptotically equivalent to $\sqrt{\frac{\pi}{2}}\sqrt{n} \approx 1.253\sqrt{n}$.*

Therefore, one measure of how close we have come to proving the Füredi conjecture is how close to this asymptotic behavior we come with the chain partitions we know how to obtain. Hsu et al. have obtained the following bounds.

**Theorem 2.20 ([13])** *For any $c > 1$, the Boolean lattice $2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains with sizes asymptotically between $0.5\sqrt{n}$ and $c\sqrt{n \log n}$.*

This is to say that there are functions asymptotically equivalent to the stated upper and lower bounds which are, respectively, an upper and a lower bound for our chain sizes.

We will use the methods we have been developing to improve on this result.

### 2.2.2 Tomon's asymptotic bounds

Tomon's results will allow us to deduce the following bounds.

**Theorem 2.21 ([21])** *For any $K \geq 2$, $2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains with sizes asymptotically between $\sqrt{2}\left(\sum_{k=2}^{K} \frac{\sqrt{\log(k)} - \sqrt{\log(k-1)}}{k}\right)\sqrt{n}$ and $\sqrt{2}\left(\sqrt{\log K} + K\sqrt{\pi}\right)\sqrt{n}$.*

Note that this result is a marked improvement on Theorem 2.20 in at least one sense, since we remove the extraneous $\sqrt{\log n}$ factor in the upper bound and so obtain chain sizes which differ from those of the Füredi partition by a constant factor. In order to prove this fact, however, we require some additional machinery.

**Definition 2.22** We extend the notion of a binomial coefficient to the complex numbers by

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)},$$

with the stipulation that $x$, $y$, and $x - y$ are not negative integers.

Observe that, since $\Gamma(k+1) = k!$ for all non-negative integers $k$, this agrees with the usual notion of a binomial coefficient for inputs on which both are defined. Moreover, since Stirling's approximation applies to the gamma function in general as well as to factorials, we can use the usual asymptotic approximations for a binomial coefficient to estimate this generalized version.

We are now in a position to verify the bounds we have claimed. The essential concept of the proof is that we will consider the monotone posets

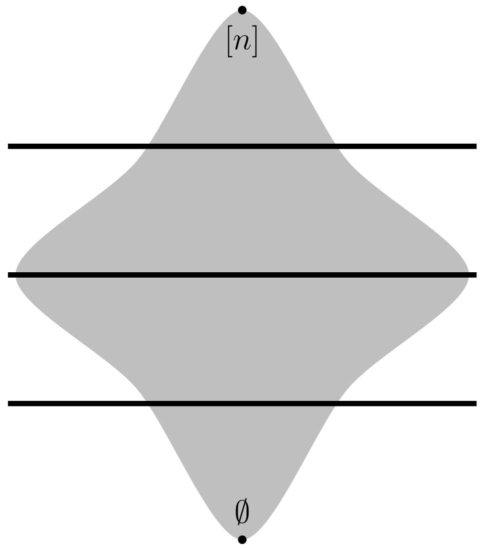


Figure 2.4: An abstract illustration of the Hasse diagram of $2^{[n]}$ for large $n$, with the four monotone subposets which we will be considering separated by horizontal lines.

defined by taking the top and bottom halves of the Boolean lattice, further split each of these at some appropriately chosen level to obtain a total of four monotone subposets, and then partition them separately, using our technique for building chains with sizes bounded below on the middle two and our technique for building chains with sizes bounded above on the top and bottom. An illustration of the bounds delimiting the four subposets is depicted in Figure 2.4.

**Proof of Theorem 2.21** Fix $K \geq 2$ and let $N$ be so large that $n \geq N$ implies that the width of $2^{[n]}$ is greater than $K$, and consider arbitrary $n \geq N$. For the sake of convenience, write $w = \binom{n}{\lfloor n/2 \rfloor}$ and

$$\alpha_K = \sqrt{2} \sum_{k=2}^{K} \frac{\sqrt{\log(k)} - \sqrt{\log(k-1)}}{k}.$$

For any $t \in \mathbb{R}$ satisfying $|t| \leq \frac{\sqrt{n}}{2}$, observe that, by some calculations with Stirling's approximation, we can obtain

$$\binom{n}{n/2 + t\sqrt{n}} \sim 2^n e^{-2t^2} \sqrt{\frac{2}{\pi n}}.$$

Since we also have $w \sim 2^n \sqrt{\frac{2}{\pi n}}$ by Lemma 2.18, we can see that

$$\binom{n}{n/2 + t\sqrt{n}} \sim we^{-2t^2}.$$

Now, for any integer $1 \leq k < w$, define $T_k$ to be the smallest natural number such that $\binom{n}{\lceil n/2 \rceil + T_k} < w/k$, which is to say that $T_k$ gives the number of levels above the middle of $2^{[n]}$ we need to progress to find a level of size less than $w/k$. Observe by our prior approximation for the binomial coefficient, the fact that $T_k/\sqrt{n}$ will satisfy the bounds needed to make the approximation valid, and the definition of asymptotic equivalence, that we can write

$$\binom{n}{\lceil n/2 \rceil + T_k} = \binom{n}{\lceil n/2 \rceil + (T_k/\sqrt{n})\sqrt{n}} = (1 + a_n)we^{-2{T_k}^2/n}$$

for some $a_n \to 0$, where the difference between $\lceil n/2 \rceil$ and $n/2$ turns out not to make a difference, as in our approximation for $w$. Therefore, we have $(1 + a_n)we^{-2{T_k}^2/n} < w/k$, giving us $k(1 + a_n) < e^{2{T_k}^2/n}$, or

$${T_k}^2 > \frac{n}{2} \log(k(1 + a_n)) = \frac{n}{2}(\log k + b_n),$$

where $b_n = \log(1 + a_n) \to 0$. Thus $T_k > \sqrt{\frac{n}{2}(\log k + b_n)}$, and we can see by the minimality of $T_k$ that in fact $T_k = \left\lfloor \sqrt{\frac{n}{2}(\log k + b_n)} \right\rfloor + 1$, since this value will satisfy the required inequality and is the smallest to do so by our strict lower bound for $T_k$. Relying on the continuity of the appropriate functions involved and the asymptotic disappearance of small error terms when divided by $\sqrt{n}$, we can rewrite this as $T_k = (1 + c_n)\sqrt{\frac{n \log k}{2}}$ for some $c_n \to 0$, or $T_k \sim \sqrt{\frac{n \log k}{2}}$.

Now let $B_0, \ldots, B_n$ denote the levels of $2^{[n]}$, let $P$ be the induced subposet on $B_{\lceil n/2 \rceil}, \ldots, B_{\lceil n/2 \rceil + T_K}$, and, for the sake of convenience, write $A_i = B_{\lceil n/2 \rceil + i}$

for all $i \in \{0, \ldots, T_K\}$. Let $f$ and $d$ be defined for $P$ as in Lemma 2.6 and observe for $2 \leq k \leq K$ and $T_{k-1} - 1 \leq j \leq T_k - k - 1$ that $f(j) = j + k$, since each of the $k$ levels $A_{j+1}, \ldots, A_{j+k}$ has size strictly less than $w/(k-1)$ and at least $w/k$. Since we must traverse each of these contiguous regions of levels while approaching $f^d(0)$ through iterated application of $f$, we can see by considering the number of levels in each region and the number of iterations needed to step through them that

$$d \geq \sum_{k=2}^{K} \left\lfloor \frac{T_k - T_{k-1} - k + 1}{k} \right\rfloor;$$

since we will be at or before the start of one of these regions $[T_{k-1}-1, T_k-k-1]$ upon exiting the last by the bounds on $f$'s output in the last, and since $f$ is monotone, this inequality is valid irrespective of $f$'s behavior on the levels between regions.

Using our asymptotic approximation for $T_k$, we can see that the right-hand side here is equal to

$$(1 + a_n)\sqrt{n/2} \sum_{k=2}^{K} \frac{\sqrt{\log(k)} - \sqrt{\log(k-1)}}{k} = (1 + a_n)\frac{\sqrt{n}}{2}\alpha_K$$

for some $a_n \to 0$.

By Lemma 2.6, there is a partition of $P$ into $w = |A_{\lceil n/2 \rceil}|$ chains, each of size at least $(1+b_n)\frac{\sqrt{n}}{2}\alpha_K$ for some $b_n \to 0$ and at most $T_K = (1+c_n)\sqrt{\frac{n \log K}{2}}$ for some $c_n \to 0$.

Now let $Q$ be the induced subposet on $B_{\lceil n/2 \rceil + T_K}, \ldots, B_n$ and observe that $Q$ is monotone with width $w' = |B_{\lceil n/2 \rceil + T_K}| \sim w/K$. Therefore, we can use Lemma 2.3 to obtain a partition of $Q$ into $w$ chains with sizes bounded above by $\frac{2|Q|}{w'} \sim \frac{2K|Q|}{w} < \frac{2K \cdot 2^{n-1}}{w} \sim K\sqrt{\frac{\pi n}{2}}$.

Merging our partitions of $P$ and $Q$ on shared elements of $B_{\lceil n/2 \rceil + T_K}$, we obtain a partition of the top half of the Boolean lattice into chains with sizes bounded below by some sequence $L_n$ and above by some sequence $U_n$ such that $L_n \sim \frac{\sqrt{n}}{2}\alpha_K$ and $U_n \sim \sqrt{\frac{n \log K}{2}} + K\sqrt{\frac{\pi n}{2}}$. By the self-duality of the Boolean lattice, we can obtain a similar partition for the levels $B_0, \ldots, B_{\lfloor n/2 \rfloor}$ and join the two partitions on shared elements, if $\lfloor n/2 \rfloor = \lceil n/2 \rceil$, or, otherwise, by using the matching between the middle two levels which covers both guaranteed by Proposition 1.36 to obtain a partition of $2^{[n]}$ into $w$ chains with sizes asymptotically bounded below by $\alpha_K\sqrt{n}$ and asymptotically bounded above by $\sqrt{2n\log K} + 2K\sqrt{\frac{\pi n}{2}} = \sqrt{2}\left(\sqrt{\log K} + K\sqrt{\pi}\right)\sqrt{n}$. ∎

Thus the theorem is correct. Using specific values of $K$ gives us numeric bounds.

**Corollary 2.23** ($K = 2$) *$2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2\rfloor}$ chains, each of size asymptotically between $0.588\sqrt{n}$ and $6.191\sqrt{n}$.*

**Corollary 2.24** ($K = 3$) *$2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2\rfloor}$ chains, each of size asymptotically between $0.690\sqrt{n}$ and $9.003\sqrt{n}$.*

**Corollary 2.25** ($K = 4$) *$2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2\rfloor}$ chains, each of size asymptotically between $0.736\sqrt{n}$ and $11.692\sqrt{n}$.*

**Corollary 2.26** ($K = 47$) *$2^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2\rfloor}$ chains, each of size asymptotically between $0.841\sqrt{n}$ and $120.587\sqrt{n}$.*

We could continue in this vein, but by this point the idea has been communicated, and the unsatisfied reader should be capable of computing further bounds independently. Moreover, it is clear that none of these results will quite give us the partition into chains of size $\sqrt{\frac{\pi}{2}}\sqrt{n}$ which we would like. Indeed, the supremum of the coefficients on the lower bounds is around 0.848 [21], well short of the desired 1.253, and the upper bound's coefficient will become arbitrarily large as this is approached. That said, these results still represent an improvement on the previously known bounds. In particular, that Tomon's techniques have allowed us to obtain bounds within a constant factor of $\sqrt{n}$ is quite remarkable, and gives hope that such techniques could be used to yield partitions with the same asymptotic behavior as the Füredi.

# Chapter 3

# Refining the Proof of Lonc's Theorem

We have, at this point, already dealt with Tomon's result on one of the two 1985 conjectures, that of Füredi. We now turn to the Sands' conjecture, which we have also previously discussed in informal terms. Properly speaking, this is no longer truly a conjecture, since its generalization by Griggs has been proven by Lonc. Therefore, we state it as a theorem.

**Theorem 3.1 (proposed in [18], proven in [17])** *Let $m$ be a positive integer. Then there exists a least positive integer $N(m)$ such that, for each $n \geq N(m)$, $2^{[n]}$ has a partition into chains of size exactly $2^m$.*

Note that, both here and in the Füredi conjecture, we are positing the existence of a partition into chains of uniform or approximately uniform size. However, the partitions we seek here differ from the Füredi in that they are not required to have exactly the minimum number of chains, and indeed it is not difficult to show that they will not in general.

Since $2^m$ will divide $|2^{[n]}| = 2^n$ for all $n \geq m$, we can see that this was, at its face, a reasonable conjecture to make on Sands' part. By the same token, it would be manifestly unreasonable to expect a partition of $2^{[n]}$ into chains of size exactly $c$ for an arbitrary positive integer $c$ and sufficiently large $n$, since $c$ will not divide the size of any Boolean lattice unless it is a power of 2. Griggs' generalization of Sands' conjecture therefore makes use of the following notion.

**Definition 3.2** Let $P$ be a poset and $c$ a positive integer. Then we define a

$c$**-partition** of $P$ to be a chain partition of $P$ such that all but at most one of the chains has size exactly $c$. If there is a chain with size not equal to $c$, we call this the **exceptional chain**.

Note that, if we have a $c$-partition of a poset with an exceptional chain, we can suppose without loss of generality that the exceptional chain has size less than $c$, since otherwise we could split it into some number of chains of size $c$ and an exceptional chain satisfying the given size constraint. Indeed, the variant of the definition which requires this is the one originally used by Griggs, but we follow Tomon in using the given version, which is slightly more flexible. Irrespective of this choice, however, the generalization can be stated as follows.

**Theorem 3.3 (proposed in [9], proven in [17])** *Let $c$ be a positive integer. Then there exists a least positive integer $N'(c)$ such that, for each $n \geq N'(c)$, $2^{[n]}$ has a $c$-partition.*

Note that, since we can break an exceptional chain up into non-exceptional chains whenever $c$ divides $|2^{[n]}|$, this does actually generalize Theorem 3.1, and in fact $N'(2^m) = N(m)$ for all $m \in \mathbb{N}$ from the definitions.

Although both Sands' conjecture and its generalization have been proven, the precise values of $N(m)$ and $N'(c)$ have yet to be determined in general. Lonc's methodology gives the following bound.

**Theorem 3.4 ([4])** *Let $c$ be a positive integer. Then $N'(c) \leq 2^{2^{36c^2}}$.*

However, it seems unlikely that this bound is close to the true value; as we have previously noted, for example, it has been shown that $N'(4) = N(2) = 9$ [8]. We will now explicate Tomon's 2016 paper *Improved bounds on the partitioning of the Boolean lattice into chains of equal size*, exploring his refinements to the previously stated bounds for $N$ and $N'$ and his further generalization of Lonc's theorem to posets beyond the Boolean lattice. As was the case in Chapter 2, although we have made slight alterations for increased precision, all the results we will cover are due to Tomon.

## 3.1 New bounds for the Sands case

Here we explore Tomon's new proof, not of Lonc's theorem in general, but of the Sands case specifically. To do so, we will need some graph-theoretic ideas not covered in Section 1.3. For the sake of brevity, and because of their widespread currency in mathematical circles, we will not treat them in detail. In what follows, let $G = (V, E)$ be a graph, and recall that we have defined all our graphs to have finitely many vertices; though many of these concepts have infinite analogues, it does not behoove us to become bogged down in the details.

**Definition 3.5** Consider $v \in V$. The **degree** of $v$, denoted $\deg_G(v)$ or simply $\deg(v)$, is the number of edges incident to $v$.

**Definition 3.6** A **path** in $G$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_n \in V$ such that, for each $i \in [n]$, $v_{i-1}v_i \in E$.

We say that $G$ is **connected** if it contains a path between any pair of distinct vertices.

**Definition 3.7** A **cycle** in $G$ is a sequence of vertices $v_0, v_1, \ldots, v_n \in V$ such that the following hold:

- For each $i \in [n]$, $v_{i-1}v_i \in E$.
- The vertices $v_1, \ldots, v_n$ are distinct.
- $v_0 = v_n$.

If $G$ does not contain any cycles, it is said to be **acyclic**.

**Definition 3.8** A connected, acyclic graph is called a **tree**. A vertex of degree 1 in such a graph is called a **leaf**.

**Definition 3.9** A **spanning tree** of $G$ is a graph $T = (V, E')$ such that $E' \subseteq E$ and $T$ is a tree.

**Theorem 3.10** *A graph has a spanning tree if and only if it is connected.*

**Definition 3.11** A **Hamiltonian path** in $G$ is a path which contains all vertices of $G$.

Note that any Hamiltonian path will define a spanning tree, since a path is connected and cannot contain a cycle.

We are now ready to take the first step toward Tomon's improved bound for $N(m)$ with the following technical lemma.

**Lemma 3.12 ([22])** *Let $P$ be a finite poset with connected comparability graph and $T = (P, E_T)$ a spanning tree of this graph. Consider positive integers $c$ and $k$, and suppose that, for each $p \in P$, we have nonnegative integers $a_p$ and $b_p$ such that the following hold:*

*(i) The sum $\sum_{p \in P} a_p + b_p$ has the same remainder as $|P \times [k]| = k|P|$ upon division by $c$.*

*(ii) For each $p \in P$, $c \deg_T(p) \leq k - a_p - b_p$.*

*(iii) For each edge $pq \in E_T$ such that $p < q$, $c(\deg_T(p) + \deg_T(q) - 1) \leq k - a_p - b_q$.*

*Then, if we let $Q = (P \times [k]) \setminus \left( \bigcup_{p \in P} \{p\} \times ([a_p] \cup ([k] \setminus [k - b_p])) \right)$, $Q$ has a partition into chains of size exactly $c$.*

The intuition for this lemma is as follows. Since $P$ is a nearly arbitrary poset, we have little hope of showing that it can be partitioned into chains of size exactly $c$. Instead, we would like to consider the result of crossing it with a sufficiently long chain — that is, of taking $k$ copies of $P$ stacked on top of one another for sufficiently large $k$. However, if we are looking for a partition into chains of size *exactly* $c$, which is to say a $c$-partition with no exceptional chain, we will still be in trouble, since there is no guarantee that $c$ will divide the size of the stacked poset.

Therefore, we choose numbers $a_p$ and $b_p$ for each $p \in P$, with the idea of removing $a_p + b_p$ of the stacked copies of $p$ to obtain a subposet $Q$ with size divisible by $c$. This need for divisibility is the reason for condition (i) in the lemma. We choose two values $a_p$ and $b_p$ instead of just one designating how many elements we should remove because we still need to know *which* copies of $p$ we should take away; our solution is to excise the $a_p$ lowest and the $b_p$ highest of them.

However, to preserve our ability to reason about the structure of $P$, we certainly do not want to remove all copies of $p$, and in fact we would like to retain some minimum number of them, determined by the degrees of $p$ and its neighbors in the spanning tree we have chosen to reason about. This is the purpose of conditions (ii) and (iii) in the lemma.

An example for a relatively uncomplicated poset with $c = 2$ and $k = 7$ is depicted in Figure 3.1. Note that here we do not specify the spanning tree for our poset's comparability graph since only one is possible.

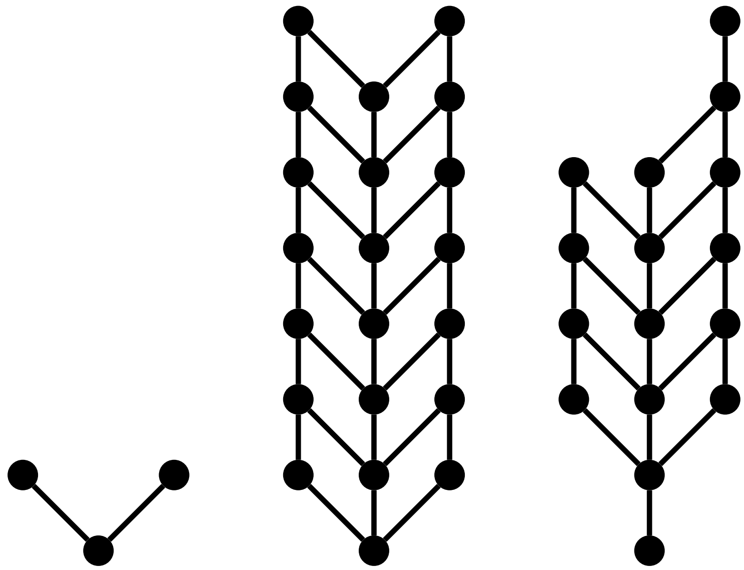

Figure 3.1: A poset $P$, $P \times [7]$, and the result $Q$ of removing some elements from $P \times [7]$ as in Lemma 3.12.

**Proof of Lemma 3.12** We proceed by induction on $|P|$.

**Base Case:** Suppose $|P| = 1$. Then $P \times [k]$ is a chain of size $k$, and hence $Q$, by condition (i), is itself a chain with size divisible by $c$. The result follows.

**Hypothesis of Induction:** Suppose $|P| > 1$ and assume that the result holds for every poset with a connected comparability graph and exactly one fewer element than $P$ to show that it holds for $P$.

**Inductive Step:** It is not difficult to show that every finite tree has a leaf. Therefore, let $u$ be a leaf of our spanning tree $T$ and $v$ the only node adjacent to it. Note that, in $P$, either $u < v$ or $v < u$.

Suppose $u < v$. Let $P'$ be the induced subposet of $P$ on $P \setminus \{u\}$ and $T'$ the tree obtained by removing $u$ and its incident edge from $T$. Note that $T'$ is a spanning tree for $P'$, since the comparability relations between the elements of $P'$ are exactly as they are in $P$ and a tree with a leaf removed is still a tree.

Now we define values $a'_p$ and $b'_p$ for each $p \in P'$. Let $a'_p = a_p$ for all $p \in P'$; that is, the number of stacked copies of any element we will remove from the bottom of $P' \times [k]$ is exactly the same as the number we were removing from $P \times [k]$. Likewise, we set $b'_p = b_p$ for every element $p$ of $P'$ except for $v$. Finally, we assign $b'_v = b_v + s$, where $s$ is the unique element of $\{0, 1, \ldots, c-1\}$ which is congruent modulo $c$ to $a_u + b_u - k$. Let $Q' = (P' \times [k]) \setminus \left(\bigcup_{p \in P'} \{p\} \times ([a'_p] \cup ([k] \setminus [k - b'_p]))\right)$.

We claim that our choice of $a'_p$ and $b'_p$ satisfies our hypotheses for $P'$, $T'$, $c$, and $k$. We verify each of the three conditions independently:

(i) Note that

$$\sum_{p \in P'} a'_p + b'_p = \left(\sum_{p \in P \setminus \{u,v\}} a_p + b_p\right) + a_v + b_v + s.$$

By our choice of $s$, this is congruent modulo $c$ to $\left(\sum_{p \in P} a_p + b_p\right) - k$, which in turn is congruent to $k|P| - k = k|P'|$ by hypothesis.

(ii) For vertices $p$ of $P'$ other than $v$, we have $c \deg_{T'}(p) \leq k - a'_p - b'_p$ by our hypotheses on the original poset, since these values are the same as their non-primed counterparts. Therefore, it suffices to show that $c \deg_{T'}(v) + a'_v + b'_v \leq k$. By our definitions for $a'_v$ and $b'_v$ and the fact that $v$ has one fewer neighbor in $T'$ than in $T$, we have $c \deg_{T'}(v) + a'_v + b'_v = c(\deg_T(v) - 1) + a_v + b_v + s < \deg_T(v) + a_v + b_v \leq k$ since $s < c$.

(iii) This condition follows by reasoning similar to that for condition (ii). Unless $v$ is one the vertices incident to the edge we have selected, the inequality we seek will follow immediately from our hypotheses. Otherwise, the decrease in $v$'s degree will either simply do us no harm or offset the increase $s$ from $b_v$ to $b'_v$.

Therefore, the hypothesis of induction gives us a partition of $Q'$ into chains of size $c$. Since we can see that $Q$ is the disjoint union of $Q'$ with $\{v\} \times ([k - b_v] \setminus [k - b_v - s])$, the set of extra copies of $v$ which we removed from $Q'$, and $\{u\} \times ([k - b_u] \setminus [a_u])$, the set of all copies of $u$ in $Q$, it now suffices to show that the union of these latter sets has a partition into chains of size $c$. Let $C = (\{u\} \times ([a_u + c - s] \setminus [a_u])) \cup$

$(\{v\} \times ([k-b_v] \setminus [k-b_v-s]))$ be the set containing our extra copies of $v$ and the $c-s$ lowest of our copies of $u$. Note that we are guaranteed that there are at least $c-s$ such copies since the total number is $k-a_u-b_u \geq c\deg_T(u) = c$ by condition (ii). We claim that $C$ is a chain; since $u < v$, it suffices to show that no element of $[a_u+c-s]\setminus[a_u]$ is greater than an element of $[k-b_v]\setminus[k-b_v-s]$. As such, we need only demonstrate $a_u + c - s \leq k - b_v - s + 1$. Rearranging this inequality, we find that we need to show $k-a_u-b_v \geq c-1 = c\deg_T(u)-1$. Since $k - a_u - b_v \geq c(\deg_T(u) + \deg_T(v) - 1) \geq c\deg_T(u) - 1$ by condition (iii), we can see that $C$ is in fact a chain. Moreover, $C$ contains $c-s$ copies of $u$ and $s$ copies of $v$, so in total it has $c$ elements.

Finally, we note that the remaining copies of $u$ lie on a chain and that there are $k - a_u - b_u - (c - s)$ of them. Since, by the definition of $s$, this value is congruent modulo $c$ to 0, we can break this chain into subchains of size exactly $c$. This completes our partition of $Q$ into such chains.

It remains to prove the result when $v < u$. The argument in this case is similar to the preceding work, except that we leave $b_v$ unchanged and instead set $a'_v = a_v + s$, also modifying our construction of the final few chains to account for this change.

The result follows by the induction. ∎

Thus, if we have a poset with connected comparability graph, crossing it with a sufficiently long chain and removing a few elements from the top and bottom of the result gives us a poset which can be partitioned into chains of size exactly $c$. Using only a very restricted case of this result, we can conclude that the product of such a poset with a sufficiently long chain has a $c$-partition (potentially with an exceptional chain).

**Lemma 3.13 ([22])** *Let $P$ be a finite poset with connected comparability graph and $T = (P, E_T)$ any spanning tree of $G(P)$. Then, if we have positive integers $c$ and $k$ such that, for every $pq \in E_T$, $c(\deg_T(p) + \deg_T(q) - 1) \leq k$, $P \times [k]$ has a $c$-partition.*

**Proof** If $|P| = 1$, the result is trivial. It is less so in the case $|P| = 2$, but, because there is only one poset on two elements which has connected comparability graph, here it is sufficient to show that, for any $k \geq c$, $[k] \times [2]$

has a $c$-partition. This follows from the observation that, if $r$ is the remainder of $k$ on division by $c$, the union of the bottom $r$ elements with [2]-coordinate 1 and the top $r$ elements with [2]-coordinate 2 form a chain, which we take to be our exceptional chain; partitioning the rest of the poset into chains of size exactly $c$ is trivial.

Now suppose $|P| \geq 3$. Let $m$ be the remainder of $|P|k$ on division by $c$, $u$ a leaf of $T$, and $v$ its sole neighbor. As before, we will suppose that $u < v$, and the $v < u$ case will be similar with the roles of the $a_p$s and $b_p$s exchanged. Let $a_p = 0$ for every $p \in P$, $b_u = m$, and $b_p = 0$ for every element $p$ of $P$ other than $u$. It is clear that these parameters satisfy condition (i) of Lemma 3.12. Moreover, condition (ii) holds; if $p \neq u$, $k - a_p - b_p = k$, and considering the inequality we have hypothesized for any edge incident to $p$ yields the result. In the case of $u$, we can see by the fact that $P$ has at least 3 elements and the fact that a tree is connected that $v$ has at least two neighbors in $T$. Thus $k \geq c(\deg_T(u) + deg_T(v) - 1) \geq c(\deg_T(u) + 1) > c\deg_T(u) + m$, so $k - a_u - b_u = k - m \geq c\deg_T(u)$.

Finally, we verify condition (iii). This follows immediately from our hypothesized inequality for every edge other than $uv$, the only one incident to $u$. For $uv$, it is sufficient to note that, since $u < v$, we must show that $c(\deg_T(u) + \deg_T(v) - 1) \leq k - a_u - b_v$, which also follows from the hypothesized inequality; our nonzero $b_u$ term never makes an impact here since $u$ is not adjacent in $T$ to any element smaller than it. Thus, by Lemma 3.12, $Q = (P \times [k]) \setminus (\{u\} \times ([k] \setminus [k-m]))$ has a partition into chains of size exactly $c$. This gives us a $c$-partition of $P$ with an exceptional chain $\{u\} \times ([k] \setminus [k-m])$ of size $m$. ∎

Hence, for any poset $P$ with a connected comparability graph, we can use the degrees of vertices in the graph's spanning tree to provide an upper bound on the smallest natural number $k$ such that $P \times [k]$ has a $c$-partition. However, a connected graph does not have a single unique spanning tree, and we can see that some trees will give better bounds than others; those where all vertices are of low degree are particularly useful. We can use this insight to obtain the following powerful result on grids.

**Theorem 3.14 ([22])** *Let $c$, $d$, and $k_1, \ldots, k_d$ be positive integers such that $k_1 \geq k_2 \geq \ldots \geq k_d$ and $k_1 \geq 3c$. Then the grid $[k_1] \times [k_2] \times \ldots \times [k_d]$ has a $c$-partition.*

**Proof** Let $P = [k_2] \times \ldots \times [k_d]$ and note that, by using induction on the dimension of the grid, it is not especially difficult to show that every grid's comparability graph has a Hamiltonian path. In particular, $P$ has a Hamiltonian path, which gives a spanning tree where every element has degree at most 2. Therefore, by Lemma 3.13, $P \times [k]$ has a $c$-partition for every $k \geq c(2+2-1) = 3c$. As such, $P \times [k_1] \cong [k_1] \times [k_2] \times \ldots \times [k_d]$ has a $c$-partition. ∎

We are now equipped to improve the previously known bound for $N(m)$, with one caveat. Because our argument will depend on the asymptotic lower bound for the chain sizes of some partition of the Boolean lattice, we will need to require that $m$ be sufficiently large for the asymptotic behavior to be relevant. We handle this difficulty as follows.

**Theorem 3.15 ([22])** *Let $L > 0$ be such that, for some $\varepsilon > 0$, the Boolean lattices $2^{[n]}$ have partitions into chains with sizes asymptotically bounded below by $(L+\varepsilon)\sqrt{n}$. Then there exists $M \in \mathbb{N}$ such that, for every $m \geq M$,*

$$N(m) \leq m + \left\lceil 9\frac{2^{2m}}{L^2} \right\rceil.$$

**Proof** For each $m \in \mathbb{N}$, define $S_m = \left\lceil 9\frac{2^{2m}}{L^2} \right\rceil$.

Observe that the Boolean lattice $2^{[n]}$ has a partition into chains of size at least $(1+a_n)(L+\varepsilon)\sqrt{n}$ for some sequence $a_n \to 0$. Therefore, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$, $2^{[n]}$ has a partition into chains of size at least $L\sqrt{n}$, since $a_n$ will eventually be so small that $(1+a_n)(L+\varepsilon) \geq L$. Let $M$ be so large that $S_M \geq N$ and consider arbitrary $m \geq M$. We can see that $S_m \geq S_M \geq N$, so $2^{[S_m]}$ has a partition into chains of size at least $L\sqrt{S_m} \geq L \cdot 3\frac{2^m}{L} = 3 \cdot 2^m$.

Thus, $2^{[m+S_m]} \cong 2^{[S_m]} \times [2]^m$ can be partitioned into the union of terms of the form $C \times [2]^m$, where $C$ is a chain of size at least $3 \cdot 2^m$. By Theorem 3.14, each of these terms has a $2^m$-partition, and since $2^m$ divides each their sizes, each such $2^m$-partition can be taken to have no exceptional chains. Thus we have a partition of $2^{[m+S_m]}$ into chains of size exactly $2^m$.

In order to show that $N(m) \leq m + S_m$, we must prove that, for every $n \geq m + S_m$, $2^{[n]}$ has a partition into chains of size $2^m$. We have just proven the case where the inequality is not strict; the other cases follow immediately since, for any $n = m + S_m + n'$, we have $2^{[n]} \cong 2^{[m+S_m]} \times 2^{[n']}$. Thus we can take all $2^{n'}$ copies of our partition of $2^{[m+S_m]}$ into chains of size $2^m$ to yield such a partition of $2^{[n]}$. ∎

Since the previously known bounds, those of Theorem 3.4, give the estimate $N(m) \leq 2^{2^{36 \cdot 2^{2m}}}$, this represents a substantial improvement for the Sands case. Note that, by our work in Section 2.2, values of $L$ satisfying our hypotheses do exist. In particular, the application of Theorem 2.21 with varying values of $K$ yields a family of possible such $L$. For example, if we use $K = 47$ and note that the coefficient on the lower bound we can achieve is actually slightly larger than the rational approximation 0.841 we gave in Corollary 2.25, we can conclude the following more concrete result.

**Corollary 3.16** *There exists $M \in \mathbb{N}$ such that, for every $m \geq M$,*

$$N(m) \leq m + \left\lceil 9\frac{2^{2m}}{0.841^2} \right\rceil \leq m + \left\lceil 12.725 \cdot 2^{2m} \right\rceil.$$

## 3.2 New bounds for the general case

Our bound for $N(m)$ can also be viewed as a bound on $N'(c)$ for special values of $c$; that is, we know that $N'(2^m) \leq \log_2(2^m) + \left\lceil 9\frac{(2^m)^2}{L^2} \right\rceil$ for any appropriate value of $L$ and sufficiently large $m$. Therefore, it is reasonable to wonder whether $N'(c)$ can be bounded above by a term on the order of $c^2$ for more general values of $c$, and, in fact, such a bound does exist. We will now build up the machinery needed to prove this assertion, starting with the following lemma on rectangles.

**Lemma 3.17 ([22])** *Let $c$, $k$, $\ell$, and $s$ be positive integers such that $k \geq \ell$, $k \geq c$, and $s \leq \frac{k+\ell}{c}$. Then the grid $P = [k] \times [\ell]$ has a $c$-partition such that the exceptional chain has size at least $(s-1)c$.*

**Proof** Let $m$ be the remainder of $|P| = k\ell$ on division by $c$ and let $M = (s-1)c + m$. Then it suffices to show that we can partition $P$ into chains such that one is of size $M$ and the rest are of size $c$.

We construct an enumeration $x_1, \ldots, x_{k\ell}$ of $P$ as follows. Envision $P$ as a grid of points in the plane, with $[k]$-entry on the horizontal axis and $[\ell]$-entry on the vertical. Then we construct our enumeration by traversing the grid in the following manner:

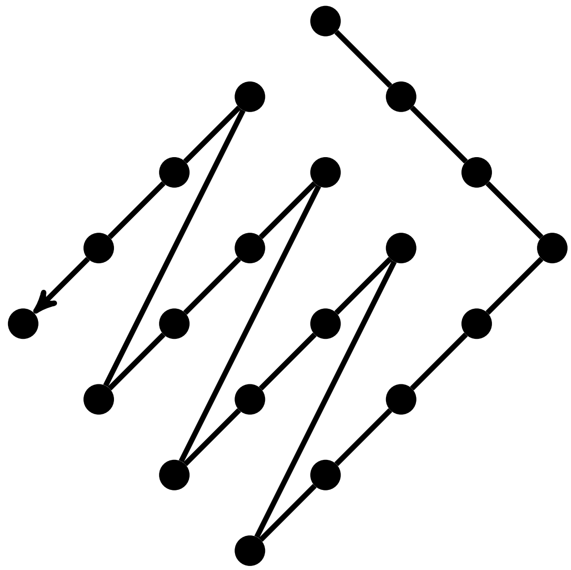

Figure 3.2: An illustration of our traversal for $[5] \times [4]$.

- Start from the upper-right corner and move element-by-element all the way down the right edge.
- Then, for each row, starting from the bottom and moving up to the top, begin from the rightmost element other than the one on the right edge which we already encountered and move left element-by-element until the end is reached.

Since our constraint on $s$ and the fact that $m < c$ together guarantee that $M$ is at most $k + \ell - 1$, the number of elements on the bottom and right edges of the grid combined, and since those elements form a chain in $P$, we can see that $x_1, \ldots, x_M$ will be a chain.

Since $M$ is equivalent modulo $c$ to $P$, the number of elements left over will be a multiple of $c$. If we partition these remaining elements by breaking the sequence $x_{M+1}, \ldots, x_{k\ell}$ into chunks of $c$ consecutive elements, we can see that the result will be a chain partition. If we start from any element $(a, b)$ for $a < k$ and keep taking subsequent elements in our enumeration, we can see that we will need to include $k + 1$ elements to prevent the set we are building from being a chain; we encounter $(a, b)$ itself, all $a - 1$ of the elements $(a - 1, b), (a - 2, b), \ldots, (1, b)$ to the left of it and in the same row,

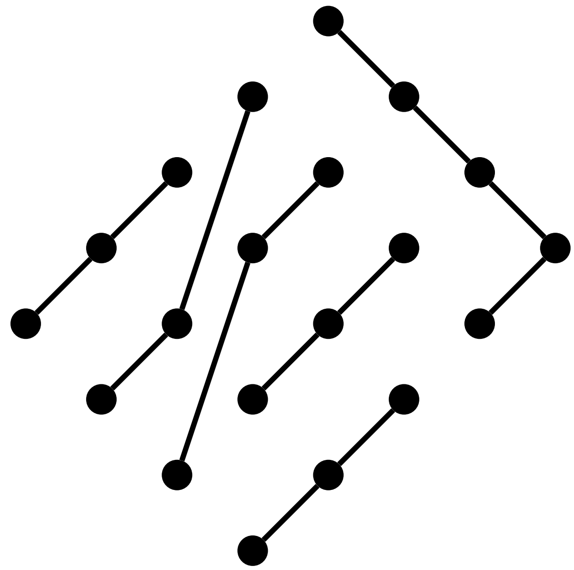

Figure 3.3: The 3-partition of $[5] \times [4]$ guaranteed by Lemma 3.17, using $s = 2$.

and all $(k-1)-(a-1)$ of the elements $(k-1, b+1), (k-2, b+1), \ldots, (a, b+1)$ one row up and neither strictly to the left of it nor in the rightmost column before we obtain an incomparable element $(a-1, b+1)$.

If, on the other hand, the element we start with is of the form $(k, b)$, we will still need at least $k + 1$ elements to generate a non-chain, since the ordering traverses the $k$ elements of the bottom row, which are each comparable to every element of the rightmost column, after finishing with this column. Since $k \geq c$, this means that all of our chunks of $c$ consecutive elements will be chains.

As such, we have partitioned $P$ into chains such that one is of size $M$ and the rest are of size $c$. $M \geq (s-1)c$, so the result follows. ∎

It is useful to have a $c$-partition with some guarantee that the exceptional chain will be long, since this means that the partition preserves more information about the relations in the original poset and hence will be easier to work with in proving other results.

We now generalize to the case of a grid of arbitrary dimension.

**Theorem 3.18 ([22])** *Let $c$, $d$, and $k_1, \ldots, k_d$ be positive integers such that*

*$k_1 \geq k_2 \geq \ldots \geq k_d$ and, for each $1 \leq i \leq d-1$,*

$$\sum_{j=1}^{i} \left\lfloor \frac{k_j}{c} \right\rfloor \geq i.$$

*Then $[k_1] \times \ldots \times [k_d]$ has a $c$-partition with exceptional chain of size at least*

$$\left( \sum_{j=1}^{d} \left\lfloor \frac{k_j}{c} \right\rfloor - (d-1) \right) c.$$

**Proof** The proof is by induction on $d$.

**Base Case:** Suppose $d = 1$. Then the result is immediate since the whole poset is a chain of size $k_1 \geq c\lfloor \frac{k_1}{c} \rfloor$.

**Hypothesis of Induction:** Suppose the result holds for some $d \geq 1$ to show that it holds for $d+1$.

**Inductive Step:** Let $k_1, k_2, \ldots, k_d, k_{d+1}$ be positive integers such that $k_1 \geq k_2 \geq \ldots \geq k_d \geq k_{d+1}$ and, for any $1 \leq i \leq d$,

$$\sum_{j=1}^{i} \left\lfloor \frac{k_j}{c} \right\rfloor \geq i.$$

Then we can apply our hypothesis of induction to obtain a $c$-partition of $[k_1] \times \ldots \times [k_d]$ such that the exceptional chain has size at least

$$\left( \sum_{j=1}^{d} \left\lfloor \frac{k_j}{c} \right\rfloor - (d-1) \right) c.$$

Let $C_1, \ldots, C_r$ be the chains of this partition, with $C_1$ the exceptional chain. Then we can partition $[k_1] \times \ldots \times [k_{d+1}]$ into rectangles by $[k_1] \times \ldots \times [k_{d+1}] = \bigcup_{i=1}^{r} C_i \times [k_{d+1}]$. For $i \geq 2$, $C_i$ is a chain of size $c$, so $C_i \times [k_d]$ can be partitioned into $k_d$ chains of size exactly $c$ in the obvious way.

Thus it suffices to obtain a $c$-partition of $C_1 \times [k_d]$ with an exceptional chain satisfying our size constraint. Since

$$|C_1| \geq \left( \sum_{j=1}^{d} \left\lfloor \frac{k_j}{c} \right\rfloor - (d-1) \right) c \geq (d - (d-1))c = c$$

by hypothesis, using Lemma 3.17 with $s = \left\lfloor \frac{|C_1|+k_{d+1}}{c} \right\rfloor$ gives us a $c$-partition of this rectangle with exceptional chain size at least

$$\left( \left\lfloor \frac{|C_1| + k_{d+1}}{c} \right\rfloor - 1 \right) c \geq \left( \left\lfloor \frac{|C_1|}{c} \right\rfloor + \left\lfloor \frac{k_{d+1}}{c} \right\rfloor - 1 \right) c.$$

Since

$$\left\lfloor \frac{|C_1|}{c} \right\rfloor \geq \sum_{j=1}^{d} \left\lfloor \frac{k_j}{c} \right\rfloor - (d-1),$$

the desired inequality for the size of the exceptional chain follows.

The induction proves the claim. ∎

We nearly ready to demonstrate an improved bound for $N'$. Before we do so, however, we must introduce one additional concept.

**Definition 3.19 ([6])** The **Gray code** is a sequence $x_1, x_2, \ldots$ of finite subsets of $\mathbb{N}$ defined recursively by $x_1 = \emptyset$ and $x_{2^m+i} = x_{2^m-(i-1)} \cup \{m+1\}$ for every nonnegative integer $m$ and $i \in [2^m]$.

This sequence was first developed by physicist Frank Gray as a way to minimize errors in transmitted binary information; the formulation we are using is due to Tomon [22]. The usefulness of the code for Gray was in the fact that each pair of consecutive subsets differs by exactly one element; we achieve this property recursively by, once we have traversed the Boolean lattice $2^{[m]}$, performing a reversed version of the same traversal in the copy of $2^{[m]}$ given by $\{S \cup \{m+1\} \mid S \in 2^{[m]}\}$ to complete our traversal of $2^{[m+1]}$. The path this traversal takes through $2^{[4]}$ is illustrated in Figure 3.4.

Indeed, the definition of the Gray code makes the following fact clear.

**Lemma 3.20** *Let $x_1, x_2, \ldots$ denote the Gray code and $m$ be a positive integer. Then the finite sequence $x_1, \ldots, x_{2^m}$ is a Hamiltonian path in the comparability graph of $2^{[m]}$.*

We will also need some properties of a particular subsequence of the Gray code defined by Tomon.

**Lemma 3.21 ([22])** *Define a sequence $\ell_1, \ell_2, \ldots$ of positive integers recursively by $\ell_1 = 2$ and, for $i \geq 2$, $\ell_i = 2^{i+1} + 1 - \ell_{i-1}$. Then $x_{\ell_1} \subset x_{\ell_2} \subset \ldots$ and, for every positive integer $i$, $x_{\ell_i - 1} \subset x_{\ell_i} \subset x_{\ell_i+1}$ if $i$ is odd and $x_{\ell_i-1} \supset x_{\ell_i} \supset x_{\ell_i+1}$ if $i$ is even.*

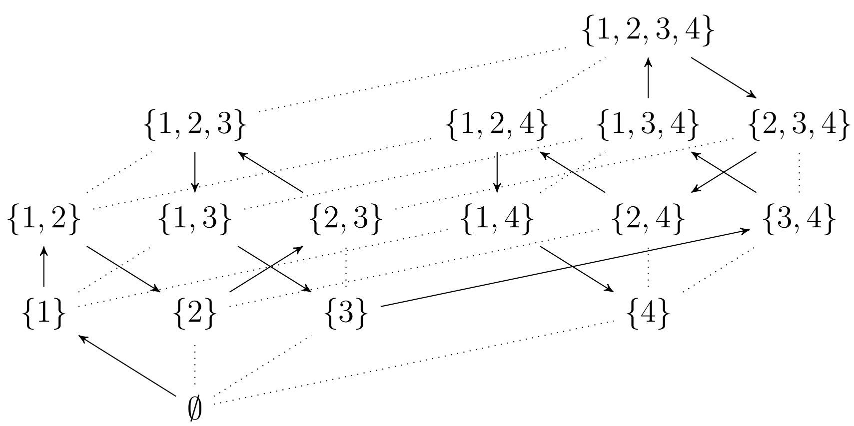


Figure 3.4: The first 16 terms of the Gray code, drawn as a traversal of the Boolean lattice $2^{[4]}$.

These results may be proven by an inductive argument; note that $\ell_i$ is defined so that $x_{\ell_i} = x_{\ell_{i-1}} \cup \{i+1\}$ and that $x_{\ell_1} = x_2 = \{1\}$ occupies a position in the Gray code such that it is above the preceding element and below its successor.

We are now ready to state and prove Tomon's improved bounds.

**Theorem 3.22 ([22])** *Let $L > 0$ be such that, for some $\varepsilon > 0$, the Boolean lattices $2^{[n]}$ have partitions into chains with sizes asymptotically bounded below by $(L+\varepsilon)\sqrt{n}$. Then there exists $C \in \mathbb{N}$ such that, for every $c \geq C$, $N'(c) \leq 5\left\lceil \frac{49c^2}{L^2} \right\rceil + 6$.*

**Proof** As in the proof of Theorem 3.15, let $N \in \mathbb{N}$ be so large that, for every $n \geq N$, $2^{[n]}$ has a partition into chains of size at least $L\sqrt{n}$, let $C$ be so large that $\left\lceil \frac{49C^2}{L^2} \right\rceil \geq N$, and consider arbitrary $c \geq C$.

Then, if we set $k_0 = \left\lceil \frac{49c^2}{L^2} \right\rceil$, we can see that $2^{[k_0]}$ has a partition into chains of size at least $L\sqrt{k_0} \geq 7c$. Fix arbitrary $k \geq k_0$ and let $t$ be the unique element of $\{3c, 3c+1, \ldots, 4c-1\}$ which is congruent modulo $c$ to $2^k$.

We will now partition $2^{[k]}$ into two posets $Q$ and $R$ so that the size of $Q$ is divisible by $c$ and $R$ is a chain. Let $x_1, x_2, \ldots$ denote the Gray code and let $\ell_1, \ell_2, \ldots$ be defined as in Lemma 3.21. Define $R = \{x_{\ell_1}, \ldots, x_{\ell_t}\}$, and note that $R$ is a subset of $2^{[k]}$, since the largest element of each $\ell_i$ is $i+1$ and $t+1 \leq 4c \leq k_0$ by virtue of the fact that $L^2$ is necessarily at most $\frac{\pi}{2}$ per Corollary 2.19. Moreover, $R$ is a chain by the first claim of Lemma 3.21.

Let $Q = 2^{[k]} \setminus R$. Note that $Q$ has a Hamiltonian path defined by removing the elements of $R$ from the sequence $x_1, \ldots, x_{2^k}$; by the second claim of Lemma 3.21, the terms of the sequence immediately before and after those we remove will be comparable, so the revised sequence remains a path.

Now consider $2^{[km]} \cong (2^{[k]})^m$ for some $m \in \{2, 3\}$ to show that it has a $c$-partition with exceptional chain size at least $5c$. Since $2^{[km]} \cong (Q \cup R)^m$, we can see that the poset can be written as the disjoint union of Cartesian products of some number of $Q$s and $R$s. Each of these terms is isomorphic to $Q^s \times R^{m-s}$ for some integer $0 \leq s \leq m$. We will consider the individual possibilities for such a term.

Suppose $s \geq 2$. Then $Q^s \times R^{m-s}$ can be partitioned into chains of size exactly $c$; to show this, it is sufficient to show that $Q^2$ has such a partition, since $Q^s \times R^{m-s}$ is the product of $Q^2$ with another poset. Since $k \geq k_0$ implies that $2^k$ can be partitioned into chains $C_1, \ldots, C_r$ of size at least $7c$, we can partition $Q$ into chains $C_1 \setminus R, \ldots, C_r \setminus R$ of size at least $7c - |R| = 7c - t > 3c$. Hence $Q^2$ has a partition into posets $Q \times (C_i \setminus R)$; since $Q$ has a spanning tree where each element has degree no more than 2, $|C_i \setminus R| \geq 3c$ gives us a $c$-partition of $Q \times (C_i \setminus R)$ by Lemma 3.13. Since $Q$ has size divisible by $c$, we can take this to have chains of size exactly $c$.

Similarly, if $s = 1$, then $m - s \geq 1$ and so $Q^s \times R^{m-s}$ is the product of $Q \times R$ with some other poset. Since $R$ is a chain of size $t \geq 3c$ and $Q$ has a Hamiltonian path, Lemma 3.13 gives us a partition of $Q \times R$ into chains of size exactly $c$ as before. Thus $Q^s \times R^{m-s}$ has such a partition as well.

Finally, we must consider the case where $s = 0$ and hence $Q^s \times R^{m-s} = R^m$. Since $|R| \geq 3c$, Theorem 3.18 easily gives us a $c$-partition of $R^m$ where the exceptional chain has size at least

$$\left( \sum_{j=1}^{m} \left\lfloor \frac{|R|}{c} \right\rfloor - (m-1) \right) c \geq (3m - (m-1))c = (2m+1)c \geq 5c.$$

Taken together, these results give us the desired partition of $2^{[km]}$.

Now consider arbitrary $n \geq 5k_0 + 6$. Since 2 and 3 are coprime, there is an integer linear combination $2n_1 + 3n_2$ of these two values equal to $n$; by the bound on $n$, there is such a combination with $n_1, n_2 \geq k_0$. As such, $2^{[n]} \cong 2^{[2n_1]} \times 2^{[3n_2]}$, and we can obtain chain partitions $C_0, C_1, \ldots, C_s$ of $2^{[2n_1]}$

and $C'_0, C'_1, \ldots, C'_{s'}$ of $2^{[3n_2]}$ such that $|C_0|, |C'_0| \geq 5c$ and all other chains have size $c$. Hence $2^{[n]}$ can be partitioned into rectangles $C_i \times C'_j$ for $0 \leq i \leq s$ and $0 \leq j \leq s'$. If either $i$ or $j$ is nonzero, $C_i \times C'_j$ has a trivial partition into chains of size exactly $c$. Moreover, since $|C_0| \geq 5c \geq c$, $C_0 \times C'_0$ has a $c$-partition by Lemma 3.17. These partitions together give us a partition for $2^{[n]}$, proving the claim. ∎

As in the case of Theorem 3.15, we can use one of our previous results in Section 2.2 to obtain a more explicit bound. For example, using Theorem 2.21 with $K = 47$ gives us the following result.

**Corollary 3.23** *There exists $C \in \mathbb{N}$ such that, for every $c \geq C$,*

$$N'(c) \leq 5\left\lceil \frac{49c^2}{0.841^2} \right\rceil + 6 \leq 5\lceil 69.28c^2 \rceil + 6.$$

Of course, all of the bounds we have provided are somewhat less useful than we might like, since they rely on the asymptotic behavior of certain families of chain partitions and hence require $c$ to satisfy some nebulously-defined lower bound. Nevertheless, they represent important progress toward understanding the behavior of $N'$ and invite further study. In particular, it may be of interest to determine the actual value of $C$ in Corollary 3.23 or a similar result and hence to ascertain whether the requirement $c \geq C$ actually requires us to exclude very many cases.

## 3.3 Generalizing Lonc's theorem

Finally, we make note of a further generalization of Lonc's theorem proven by Tomon [22]. Although it might seem natural to try to generalize to the case of an arbitrary unimodal normalized matching poset, as Tomon has proposed for the Füredi conjecture, this is not possible naively since we require some family of posets parameterized by a natural number in order to state the claim. Instead, we recall that $2^{[n]} \cong [2]^n$ and hence generalize to the powers of other posets as follows.

**Theorem 3.24 ([22])** *Let $P$ be a finite poset with connected comparability graph and $c$ a positive integer. Then there is a least positive integer $N(P, c)$ such that, for each $n \geq N(P, c)$, $P^n$ has a $c$-partition.*

Thus Lonc's theorem is really the simplest case of a much more general fact about powers of posets. Since our focus is on the Boolean lattice, and the proof of this result is actually quite similar to that of Theorem 3.22, we will not go into detail; for the curious, the argument may be found in Tomon's original paper [22]. We would, however, like to note that the proof uses the following result, which is interesting in its own right.

**Theorem 3.25 ([22])** *Let $P$ be a finite poset whose comparability graph contains no isolated elements. Then there exists a constant $c_P > 0$ such that $P^n$ can be partitioned into chains with sizes asymptotically bounded below by $c_P\sqrt{n}$.*

# Chapter 4

# Conclusions and Future Work

This concludes our study of Tomon's results. The methodology he has introduced is groundbreaking, far outdoing the previously known approaches to the problems which we have examined. However, much work remains to be done in both cases.

Although Tomon's bounds of Section 2.2 are quite good, they, as we have noted, still fall short of the asymptotic behavior of the Füredi partition. Moreover, there is reason to believe that a straightforward refinement of such methods cannot be capable of rectifying this. We have seen that the application of Theorem 2.21 involves a trade-off between the quality of the upper and lower bounds, depending on the choice of $K$, and this limitation seems fundamental to Tomon's approach; if we bifurcate a half of the Boolean lattice, partition each part separately into chains, and then merge the partitions, there will always be variation among the chain sizes on the order of the rank of the portion of the half-lattice which does not contain the lattice's largest level. Therefore, it seems likely that yet more sophisticated techniques which consider the Boolean lattice as a whole instead of piecemeal, or at least use a deeper method of joining the partitions of the two half-lattices, will ultimately be needed.

In the case of Lonc's theorem, Tomon's bounds are very promising. As we have mentioned, we believe it worthy of study whether the requirement that the chain sizes we are searching for be sufficiently large in order for the bounds to apply really makes much of an impact. In addition, the question of whether these bounds are the best possible has yet to be answered. Tomon does not believe so; rather, he proposes the following conjecture.

**Conjecture 4.1 ([22])** *Let $c$ be a positive integer. Then $N'(c) = n_0 + \Lambda(c)$, where $\Lambda(c)$ is a term on the order of $\log c$ and $n_0$ is the smallest positivie integer such that*

$$\frac{2^{n_0}}{\binom{n_0}{\lfloor n_0/2 \rfloor}} > c.$$

The role of the $n_0$ term here is to guarantee that the chains of the Füredi partition are at least as large as those we are trying to find; if this condition is not met, it is relatively easy to see that we have no hope of obtaining a $c$-partition. However, no proof of this result has yet been discovered.

Thus these areas of inquiry remain quite open. For our part, we believe that the most promising route to the Füredi partition and to precise values of $N'(c)$ is through further study of the Griggs conjecture [9], to which we alluded in the our introduction. In particular, the set of possible collections of chain sizes for a partition proposed by Griggs itself forms a poset, the structure of which does not seem to have been studied in detail; a better understanding of this object and the relationship it bears to the Boolean lattice has the potential to be very useful in our efforts to obtain concrete chain partitions corresponding to each of its elements.

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